Vanishing topology of codimension 1 multi-germs over $\mathbb{R}$ and $\mathbb{C}$ *

T. Cooper, D. Mond and R. Wik Atique

February 1, 2008

Abstract

We construct all $\mathcal{A}_e$-codimension 1 multi-germs of maps $(k^n, T) \to (k^p, 0)$, with $n \geq p-1$, (n,p) nice dimensions, $k = \mathbb{C}$ or $\mathbb{R}$, by augmentation and concatenation operations, starting from mono-germs ($|T| = 1$) and one 0-dimensional bi-germ. As an application, we prove general statements for multi-germs of corank ≤ 1: every one has a real form with real perturbation carrying the vanishing homology of the complexification, every one is quasihomogeneous, and when $n = p-1$ every one has image Milnor number equal to 1 (this last a result already known when $n \geq p$).

1 Introduction

In this paper we investigate the topology of the discriminant of stable perturbations f_t of multi-germs $f: (k^n, S) \to (k^p, 0)$ with $n \geq p-1$, where $S \subset k^n$ is a finite set, and where $k = \mathbb{R}$ or $\mathbb{C}$. When $n = p-1$ 'discriminant' of course means 'image'.

When $k = \mathbb{C}$, the discriminant $D(f_t)$ has the homotopy type of a wedge of $(p-1)$-spheres ([4],[20]). The number of these spheres is called the **discriminant Milnor number** when $n \geq p$ and the **image Milnor number** when $n = p-1$, and denoted μ_Δ and μ_I respectively. When $n \geq p$ and (n,p) are in Mather's range of nice dimensions ([17]), it is known ([4]) that $\mu_\Delta(f)$ and the $\mathcal{A}_e$- codimension of f satisfy the Milnor-Tjurina relation:

$$\mu_\Delta(f) \geq \mathcal{A}_e\text{-codimension}(f)$$

with equality if f is weighted homogeneous in some coordinate system. In case $n = p-1$, the same relation, with μ_I in place of μ_Δ, is only known to hold when $n = 1$ ([21]) and $n = 2$ ([11],[20]). Nevertheless there is evidence that it holds in higher dimensions (see e.g. [10]):

*Subject Classification: *Primary* 32S05, 32S30, 14B05, *Secondary* 14P25, 32S70

Conjecture I This relation holds in all nice dimensions $(n, n+1)$.

Here we are concerned with this conjecture, and also with another: suppose that $g : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ is a real analytic map germ of finite $\mathcal{A}$-codimension, with a stable perturbation g_t. Suppose also that the complexification $g_{\mathbb{C},t}$ of g_t is a stable perturbation of the complexification $g_{\mathbb{C}}$ of g. We say that g_t is a **good real perturbation** of g if $\operatorname{rank} H_{p-1}(D(g_t);\mathbb{Z}) = \operatorname{rank} H_{p-1}(D(g_{\mathbb{C},t});\mathbb{Z})$ (in which case the inclusion of real in complex induces an isomorphism on the vanishing homology of the discriminant).

Conjecture II For every $\mathcal{A}_e$-codimension 1 equivalence class of map-germs in the nice dimensions, there exists a real form with a good real perturbation. That is, the vanishing topology of all codimension 1 complex singularities is 'visible over $\mathbb{R}$'.

For maps $\mathbb{C}^2 \to \mathbb{C}^3$ there are five codimension 1 equivalence- classes (see Figure 1 on page 10); for maps $\mathbb{C}^3 \to \mathbb{C}^4$ there are eight, and for maps $\mathbb{C}^4 \to \mathbb{C}^5$ there are eleven.

Conjecture II is known to hold for mono-germs of maps $\mathbb{C}^n \to \mathbb{C}^p$ (with $n \geq p$ and (n, p) nice dimensions) of corank 1 ([19]). It also holds for (mono- and multi-) germs of maps $\mathbb{C}^2 \to \mathbb{C}^3$ ([6]; Goryunov's diagrams of good real perturbations are reproduced in Figure 1 below). Every real germ $\mathbb{C} \to \mathbb{C}^2$ has a good real perturbation ([1],[8]), but once $n > 1$, map-germs $\mathbb{C}^n \to \mathbb{C}^{n+1}$ with good real perturbations become the exception ([14]).

Our main results here provide evidence for both conjectures. We show

Theorem 7.2 Every multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ of corank 1 and $\mathcal{A}_e$-codimension 1 has $\mu_I(f) = 1$.

Theorem 7.3 Every $\mathcal{A}$-equivalence class of multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ $(n \geq p - 1, (n, p)$ nice dimensions) of corank 1 and $\mathcal{A}_e$-codimension 1 has a real form with a good real perturbation.

We prove both of these theorems first for 'mono-germs' ($|S| = 1$) (in Section 4) and then by an inductive procedure which constructs codimension 1 multi-germs from simpler ingredients. This procedure yields an inductive classification of multi-germs of codimension 1. In Section 5 we show that all codimension 1 multi-germs can be constructed from codimension 1 multi-germs with fewer branches and in a lower dimension, and from trivial unfoldings of Morse singularities (in case $n \geq p$) or immersions (in case $p = n + 1$) by means of three standard operations. These are *augmentation*, described in Section 2, and two *concatenation operations*, described in Section 3.

We feel that these operations, of augmentation and concatenation, are themselves of independent interest. They can be seen at work, generating the lists of $\mathcal{A}_e$-codimension 1 germs from surfaces to 3 space, and from surfaces to surfaces, in Figures 1 and 2, on page

10. See also Figure 3, on page 13.

We end this introduction with an elementary lemma which nevertheless highlights an important property of codimension 1 germs.

Lemma 1.1 *If $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a germ of $\mathcal{A}_e$-codimension 1, then any stable unfolding of f is $\mathcal{A}_e$-versal.*

Proof Let $F(x,u) = (f_u(x), u)$ be a d-parameter unfolding of f. By [15] XV 2.1, F is stable iff

$$T\mathcal{A}_e f + \mathcal{O}_p\{\partial f_u/\partial u_1|_{u=0}, \ldots, \partial f_u/\partial u_d|_{u=0}\} = \theta(f).$$

Since $T\mathcal{A}_e f$ is an $\mathcal{O}_p$-module, we therefore cannot have $\partial f_u/\partial u_i|_{u=0} \in T\mathcal{A}_e f$ for all i. Hence for some i, $T\mathcal{A}_e f + \mathbb{C}\{\partial f_u/\partial u_i|_{u=0}\} = \theta(f)$, and F is versal. □

The results in this paper concerning maps $\mathbb{C}^n \to \mathbb{C}^{n+1}$, and the results of Sections 2 and 5, were first proved in the Ph.D. thesis ([2]) of the first author.

2 Augmentations

Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a multi-germ of $\mathcal{A}_e$-codimension 1 where S is a finite subset of $\mathbb{C}^n$. Let

$$\begin{aligned} F : (\mathbb{C} \times \mathbb{C}^n, \{0\} \times S) &\to (\mathbb{C} \times \mathbb{C}^p, (0,0)) \\ (\lambda, x) &\mapsto (\lambda, f_\lambda(x)) \end{aligned}$$

be an $\mathcal{A}_e$-versal unfolding of f. Define $A_F : (\mathbb{C} \times \mathbb{C}^n, \{0\} \times S) \to (\mathbb{C} \times \mathbb{C}^p, (0,0))$ by $A_F(\lambda, x) = (\lambda, f_{\lambda^2}(x))$.

Proposition 2.1 *The $\mathcal{A}$-equivalence class of A_F is independent of the choice of miniversal unfolding F of f. It depends only on the $\mathcal{A}$-equivalence class of f.*

Proof Let $F(t,x) = (t, f_t(x))$ and $G(s,x) = (s, g_s(x))$ be two 1-parameter versal unfoldings of f. From the definition of versality it follows immediately that there exist diffeomorphisms $\Phi(t,x) = (t, \phi_t(x))$ and $\Psi(t,y) = (t, \psi_t(y))$ and a base-change diffeomorphism $\alpha : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $\alpha^*(F)(t,x) = \Psi \circ G \circ \Phi$ (where $\alpha^*(F)$ is the unfolding $(t,x) \mapsto (t, f_{\alpha(t)}(x))$. An easy calculation shows that there exists $\beta : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ (also invertible) such that $\alpha(t^2) = \beta(t)^2$; now writing $A_\Phi(t,x) = (t, \phi_{t^2}(x))$ and $A_\Psi(t,y) = (t, \psi_{t^2}(y))$ we have $\beta^*(A_F) = A_\Psi \circ A_G \circ A_\Phi$.

Equivalence of germs entails equivalence of their miniversal unfoldings, so the second statement follows. □

We shall write Af for the $\mathcal{A}$-equivalence class of A_F. We call Af the **augmentation** of f and say that a multi-germ is an augmentation if and only if it is the augmentation of some multi-germ f. A multi-germ that is not an augmentation is called **primitive**.

Example 2.2 The five $\mathcal{A}_e$-codimension 1 multi-germs from $\mathbb{C}^2$ to $\mathbb{C}^3$ are:

I. S_1 (the birth of two umbrellas).

II. The non-transverse contact of two immersed sheets.

III. The intersection of three immersed sheets which are pairwise transverse, but with each one having first order tangency to the intersection of the other two.

IV. A cross-cap meeting an immersed plane.

V. A quadruple intersection.

IV and V are primitive. I is the augmentation of the cusp $t \mapsto (t^2, t^3)$, II is the augmentation of a tacnode (two curves simply tangent at a point), which itself is the augmentation of the map from two copies of $\mathbb{C}^0$ to $\mathbb{C}$ sending both points to $0 \in \mathbb{C}$, and III is the augmentation of three lines meeting pairwise transversely at a point.

Pictures of the images of good real perturbations of these germs, showing the process of augmentation, are shown in Figure 1 on page 10.

Example 2.3 The five $\mathcal{A}_e$-codimension 1 multi-germs from $\mathbb{C}^2$ to $\mathbb{C}^2$ are:

I. The lips: $(x, y) \mapsto (x, y^3 + x^2 y)$;

II. The swallowtail: $(x, y) \mapsto (x, y^4 + xy)$;

III. The fold tacnode - a bi-germ consisting of two folds whose discriminant curves have a simple tangency;

IV. The fold triple-point: a tri-germ consisting of three folds whose discriminants meet pairwise transversely at a point;

V. A bi-germ consisting of a fold and a cusp, with the discriminant of the fold transverse to the limiting tangent line to the discriminant of the cusp.

I is the augmentation of $y \mapsto y^3$; II is primitive; III is the augmentation of the bi-germ consisting of the two branches $x \mapsto x^2$ and $y \mapsto y^2$; IV and V are both primitive.
Pictures of the discriminants of good real perturbations of these germs are shown in Figure 2 on page 10.

Theorem 2.4 *Af has $\mathcal{A}_e$-codimension 1.*

Proof We use Damon's theory of $\mathcal{K}_V$-equivalence (see for example [3]).

The diagram

$$\begin{array}{ccc} \mathbb{C}^{n+1} & \xrightarrow{F} & \mathbb{C}^{p+1} \\ \uparrow \mathrm{id} & & \uparrow \gamma \\ \mathbb{C}^{n+1} & \xrightarrow{A_F f} & \mathbb{C}^{p+1} \end{array}$$

where $\gamma(\delta, y) = (\delta^2, y)$, is a transverse fibre square. Therefore the $\mathcal{A}_e$-codimension of $A_F f$ is equal to the $\mathcal{K}_{D(F),e}$-codimension of γ where $D(F)$ is the discriminant of F.

But the diagram

$$\begin{array}{ccc} \mathbb{C}^{n+1} & \xrightarrow{F} & \mathbb{C}^{p+1} \\ \uparrow i_1 & & \uparrow i_2 \\ \mathbb{C}^n & \xrightarrow{f} & \mathbb{C}^p \end{array}$$

where i_1 and i_2 are inclusions, is also a transverse fibre square. So the $\mathcal{K}_{D(F),e}$-codimension of i_2 is equal to the $\mathcal{A}_e$-codimension of f and therefore is 1.

Because i_2 is a standard coordinate immersion, an easy calculation shows

$$N\mathcal{K}_{D(F),e}i_2 = \frac{\mathcal{O}_p}{d\lambda(i_2^*(\mathrm{Der}(\log D(F))))}$$

(where $d\lambda(i_2^*(\mathrm{Der}(\log(D(F))))$ is the module consisting of the coefficients of $\partial/\partial\lambda$ of the elements of $i_2^*(\mathrm{Der}(\log D(F)))$. A similar calculation gives

$$N\mathcal{K}_{D(F),e}\gamma = \frac{\mathcal{O}_{p+1}}{d\lambda(\gamma^*(\mathrm{Der}(\log D(F)))) + (\delta)}$$

where the (δ) in the denominator comes from $\partial\gamma/\partial\delta$. Clearly $N\mathcal{K}_{D(F),e}i_2$ and $N\mathcal{K}_{D(F),e}\gamma$ are isomorphic. $\square$

The following result is a partial converse.

Proposition 2.5 *Suppose that $G(\lambda, x) = (\lambda, g_\lambda(x))$ is a one-parameter stable unfolding of a multi-germ $g = g_0$ and suppose that $h(\lambda, x) = (\lambda, g_{\lambda^2}(x))$ has $\mathcal{A}_e$-codimension 1. Then g has $\mathcal{A}_e$-codimension 1 and G is a versal unfolding of g. Thus h is the augmentation of g.*

Proof It is immediate from the calculation in the proof of 2.4 that g has $\mathcal{A}_e$-codimension 1. Versality of G now follows by 1.1. $\square$

Given a stable map $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ let Pf (the 'prism' on f) be the trivial 1-parameter unfolding of f. We shall say that a map-germ is a **prism** if it is $\mathcal{A}$-equivalent to Pg for some germ g.

An easy calculation with tangent spaces shows

Proposition 2.6 *Let $F(\lambda, x) = (\lambda, f_\lambda(x))$ be an $\mathcal{A}_e$-versal unfolding of an $\mathcal{A}_e$-codimension 1 multi-germ f. Then $G(\mu, \lambda, x) = (\mu, \lambda, f_{\lambda^2+\mu}(x))$ is an $\mathcal{A}_e$-versal unfolding of $g = A_F f$.* $\square$

Since $G(\mu, \lambda, x) = (\mu, \lambda, f_{\lambda^2+\mu}(x))$ is an unfolding of $F(\mu, x) = (\mu, f_\mu(x))$ and F is stable then G is $\mathcal{A}$-equivalent to PF. Therefore if a multi-germ is an augmentation, its miniversal unfolding is a prism. The converse is also true:

Theorem 2.7 *Let g be a multi-germ of $\mathcal{A}_e$-codimension 1 and suppose that the miniversal unfolding G of g is a prism. Then g is an augmentation.*

Proof There is a unique natural number ℓ and a stable multi-germ h, unique up to $\mathcal{A}$-equivalence, such that $G(\lambda, x) = (\lambda, g_\lambda(x))$ is $\mathcal{A}$-equivalent to $P^\ell h$ and h is not a prism.

We have the following commutative diagram

$$\begin{array}{ccc}
\mathbb{C}^n, S & \xrightarrow{g} & \mathbb{C}^p, 0 \\
\downarrow & & \downarrow i \\
\mathbb{C} \times \mathbb{C}^n, \{0\} \times S & \xrightarrow{(\lambda, g_\lambda(x))} & \mathbb{C} \times \mathbb{C}^p, (0,0) \\
\downarrow \phi & & \downarrow \psi \\
\mathbb{C}^\ell \times \mathbb{C}^{n+1-\ell}, \{0\} \times S' & \xrightarrow{id_{\mathbb{C}^\ell} \times h} & \mathbb{C}^\ell \times \mathbb{C}^{p+1-\ell}, (0,0) \\
\downarrow & & \downarrow \pi \\
\mathbb{C}^{n+1-\ell}, S' & \xrightarrow{h} & \mathbb{C}^{p+1-\ell}, 0
\end{array}$$

where i is the standard inclusion, ϕ and ψ are diffeomorphisms, π is the natural projection and S' is a subset of $\mathbb{C}^{n+1-\ell}$ of the same cardinality as S. Each of the three squares of the diagram is a transverse fibre square so the outside rectangle is a transverse fibre square as well. The $\mathcal{A}_e$-codimension of g is equal to the $\mathcal{K}_{D(h),e}$-codimension of $\pi \circ \psi \circ i$ where $D(h)$ is the discriminant of h. Since h is stable it is Thom transversal so any vector field in $\mathrm{Der}(\log D(h))$ lifts, by 6.14 of [12]. Since h is not a prism, $\mathrm{Der}(\log D(h)) \subseteq m_{p+1-\ell}\theta(p+1-\ell)$. So,

$$T\mathcal{K}_{D(h),e}(\pi \circ \psi \circ i) \subseteq T\mathcal{K}_e(\pi \circ \psi \circ i)$$

and the $\mathcal{K}_e$-codimension of $\pi \circ \psi \circ i$ is 0 or 1. It cannot be 0, as this would make $\pi \circ \psi \circ i$ a submersion and g stable. Therefore $\pi \circ \psi \circ i$ is a quadratic singularity, $\mathcal{A}$-equivalent to

$$(y_1, \ldots, y_p) \overset{\gamma}{\mapsto} (y_1, \ldots, y_{p-\ell}, \sum_{i=p+1-\ell}^{p} y_i^2)$$

Let Φ and Ψ be germs of diffeomorphisms such that $\Psi \circ (\pi \circ \psi \circ i) = \gamma \circ \Phi$.

Let $\pi_{p+1-\ell} : \mathbb{C}^{p+1-\ell} \to \mathbb{C}$ be projection onto the last coordinate. Then $d(\pi_{p+1-\ell} \circ \Psi \circ (\pi \circ \psi \circ i))(0) = 0$ and since h is transverse to $\pi \circ \psi \circ i$, $d(\pi_{p+1-\ell} \circ \Psi \circ h)(S') \neq 0$. It follows that for λ near 0, $(\pi_{p+1-\ell} \circ \Psi \circ h)^{-1}(\lambda) \cong \mathbb{C}^{n-\ell}$ and $(\pi_{p+1-\ell} \circ \Psi)^{-1}(\lambda) \cong \mathbb{C}^{p-\ell}$.

Define $h_\lambda = h|_{(\pi_{p+1-\ell} \circ \Psi \circ h)^{-1}(\lambda)} : \mathbb{C}^{n-\ell} \to \mathbb{C}^{p-\ell}$. Then $h(\lambda, x) = (\lambda, h_\lambda(x))$ is an unfolding of h_0. Since the outside rectangle of the above diagram is a transverse fibre square, g is $\mathcal{A}$-equivalent to the germ $(\lambda_1, \ldots, \lambda_\ell, x) \mapsto (\lambda_1, \ldots, \lambda_\ell, h_{\sum_{i=1}^\ell \lambda_i^2}(x))$. Therefore, g is an augmentation by Proposition 2.5. □

3 Concatenation

In this section we describe two basic operations, by which we "concatenate" stable unfoldings of (multi-) germs to create new multi-germs. There is no reason to require purity of dimension in multi-germs, and we allow different branches to have domains of different

dimension. We therefore will not distinguish in our notation between image Milnor number and discriminant Milnor number: both will be denoted μ_Δ. In what follows it will be useful to use the notation $\{f, g\}$ for the germ obtained by putting together germs f and g with the same target.

Throughout this section we assume that we are in the nice dimensions; thus, every stable unfolding $(f_\lambda(x), \lambda)$ of a germ f_0 is a "stabilisation", in the sense that for almost all λ, f_λ is stable.
The first concatenation operation is monic: from a multi-germ with m branches we get a multi-germ with $m+1$ branches, in which the extra branch is a fold or an immersion.

Theorem 3.1 *Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a map-germ of finite $\mathcal{A}_e$-codimension with a stable unfolding F on the single parameter t, let $0 \le k \in \mathbb{Z}$ and let $g : (\mathbb{C}^p \times \mathbb{C}^k, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ be the fold map $(y, v) \mapsto (y, \sum_{j=1}^k v_j^2)$. Then*

1. $$\mathcal{A}_e - codim(g^*(F)) = \mathcal{A}_e\text{-}codim(f) = \mathcal{A}_e\text{-}codim(\{F, g\})$$

2. $$\mu_\Delta(g^*(F)) = \mu_\Delta(\{F, g\}) = \mu_\Delta(f)$$

3. *both $g^*(F)$ and $\{F, g\}$ have 1-parameter stable unfoldings.*

Proof (1) Let $i : (\mathbb{C}^p, 0) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ be the standard inclusion inducing f from F. By Damon's theorem the $\mathcal{A}_e$-codimension of f is equal to the vector-space dimension of $N\mathcal{K}_{D(F),e}i := \theta(i)/ti(\theta_{\mathbb{C}^p}) + i^*(\mathrm{Der}(\log D(F)))$. As i is an immersion, projecting to the last component gives an isomorphism $N\mathcal{K}_{D(F),e}i \simeq \mathcal{O}_{\mathbb{C}^p,0}/dt(i^*(\mathrm{Der}(\log D(F))))$. Again by Damon's theorem, the $\mathcal{A}_e$-codimension of $g^*(F)$ is equal to the dimension of $N\mathcal{K}_{D(F),e}g$; since $tg(\theta_{\mathbb{C}^p\times\mathbb{C}^k}) = \sum_{\ell=1}^p \mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^k} \cdot \partial/\partial y_\ell + \sum_{j=1}^k \mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^k} \cdot v_j\partial/\partial t$, it follows, again by projecting to the last component, that

$$N\mathcal{K}_{D(F),e}g \simeq \mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^k,0}/(v_1, \cdots, v_k) + dt(g^*(\mathrm{Der}(\log D(F))));$$

this in turn is isomorphic to $\mathcal{O}_{\mathbb{C}^p,0}/dt(i^*(\mathrm{Der}(\log D(F))))$, and thus to $N\mathcal{K}_{D(F),e}i$. This proves the first equality in (1).

To prove the second equality in (1), we use the exact sequence

$$0 \to \frac{\theta(g)}{tg(\theta_{\mathbb{C}^p\times\mathbb{C}^k}) + \omega g(\mathrm{Der}(\log D(F)))} \to N\mathcal{A}_e\{F, g\} \to N\mathcal{A}_e F \to 0$$

which results from the fact that $\mathrm{Der}(\log D(F))$ is the kernel of $\overline{\omega}F : \theta_{\mathbb{C}^p\times\mathbb{C}} \to \theta(F)/tF(\theta_{\mathbb{C}^n\times\mathbb{C}})$. Since F is stable, $N\mathcal{A}_e\{F, g\}$ is isomorphic to $\theta(g)/tg(\theta_{\mathbb{C}^p\times\mathbb{C}^k}) + \omega g(\mathrm{Der}(\log D(F)))$. This in turn is isomorphic to $\mathcal{O}_{\mathbb{C}^p\times\mathbb{C}^k,0}/(v_1, \cdots, v_k) + dt(\omega g(\mathrm{Der}(\log D(F)))$, by projection to the last component, and thus, evidently, to $\mathcal{O}_{\mathbb{C}^p,0}/dt(i^*(\mathrm{Der}(\log D(F)))$, i.e. to $N\mathcal{K}_{D(F),e}i$.

(2) For $\lambda \neq 0$, the map g_λ defined by $g_\lambda(y,v) = (y, \sum v_j^2 + \lambda)$ is logarithmically transverse to $D(F)$. Thus $g_\lambda^*(F)$ is a stable perturbation of $g^*(F)$. Its discriminant is $g_\lambda^{-1}(D(F))$. There are now two cases, $k > 0$ and $k = 0$.

If $k > 0$, $g_\lambda^{-1}(D(F))$ fibres over $D(F)$ with typical fibre diffeomorphic to the Milnor fibre X_g of g, and contractible fibres over the points of $D(F) \cap D(g_\lambda)$. Since $D(F)$ itself is contractible, it follows that $g_\lambda^{-1}(D(F))$ is homotopy- equivalent to the space obtained from $D(F) \times X_g$ by gluing in a k-ball to each fibre over $D(F) \cap D(g_\lambda)$ to kill its homotopy. A Mayer-Vietoris argument now shows that the rank of $H_{p+k-1}(g_\lambda^{-1}(D(F)))$ is equal to the rank of $H_{p-1}(D(F) \cap D(g_\lambda))$. Since $\{F, g_\lambda\}$ is a stable perturbation of $\{F, g\}$, a second Mayer-Vietoris argument shows that $H_p(D(F) \cup D(g_\lambda)) \simeq H_{p-1}(D(F) \cap D(g_\lambda))$; thus

$$\mu_\Delta(g^*(F)) = \text{rank } H_{p+k-1}(g_\lambda^{-1}(D(F))) = \text{rank } H_p(D(F) \cup D(g_\lambda)) = \mu_\Delta\{F, g\}.$$

The second equality of (2) follows from the fact that $D(g_\lambda) = i_\lambda(\mathbb{C}^p) = D(i_\lambda^*(F))$, where $i_\lambda : \mathbb{C}^p \to \mathbb{C}^p \times \mathbb{C}$ is defined by $y \mapsto (y, \lambda)$. For i_λ is logarithmically transverse to $D(F)$, and thus $i_\lambda^{-1}(D(F))$ (for $\lambda \neq 0$) is the discriminant of a stable perturbation $i_\lambda^*(F)$ of f.

If $k = 0$, the situation is much simpler: $g_\lambda^{-1}(D(F))$ is diffeomorphic to $D(g_\lambda) \cap D(F)$, and the assertion is proved by a similar Mayer-Vietoris argument.

(3) The unfolding $G = (g_\lambda, \lambda)$ of g induces from $F \times \mathrm{id}_{\mathbb{C}}$ a stable unfolding of $g^*(F)$, since it is logarithmically transverse to $D(F) \times \mathbb{C}$. The unfolding $\{F \times \mathrm{id}_{\mathbb{C}}, G\}$ of $\{F, g\}$ is stable, since the analytic stratum $\mathbb{C}^p \times \mathbb{C} \cdot (1,1)$ of G is tranverse to the analytic stratum of $F \times \mathrm{id}_{\mathbb{C}}$. $\square$

In particular, if the germ f satisfies Conjecture 1, then so does $\{F, g\}$. In fact, our proof of 3.1 shows that the same goes for the existence of good real perturbations (Conjecture 2, in the case of map-germs of codimension 1).

Theorem 3.2 *If f has a good real perturbation then so does $\{F, g\}$, and vice versa.*

Proof Replace $\mathbb{C}$ by $\mathbb{R}$ everywhere in the topological part of the proof of 3.1. The Mayer-Vietoris argument shows that $\operatorname{rank} H_p(D_{\mathbb{R}}(F) \cup D_{\mathbb{R}}(g_\lambda)) = \operatorname{rank} H_{p-1}(D_{\mathbb{R}}(F) \cap D_{\mathbb{R}}(g_\lambda)) = \operatorname{rank} H_{p-1}(D(f_t))$, so that if either side has, for $t > 0$ or for $t < 0$, rank equal to the rank of the homology of the complexification, then so, by 3.1, does the other. $\square$

Theorem 3.3 *Suppose that the germ f of Theorem 3.1 has $\mathcal{A}_e$- codimension 1. Then up to $\mathcal{A}$-equivalence, the bi-germ $h = \{F, g\}$ obtained is independent of the choice of stable unfolding F.*

Proof Any stable 1-parameter unfolding of f is also $\mathcal{A}_e$-versal. Thus, given two such, F' and F'', by the semi-uniqueness of mini-versal unfoldings there are a diffeomorphism

$\alpha : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ and unfoldings of the identity $\phi : (\mathbb{C}^n \times \mathbb{C}, S \times \{0\}) \to (\mathbb{C}^n \times \mathbb{C}, S \times \{0\})$ and $\Psi : (\mathbb{C}^p \times \mathbb{C}, \{0\} \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}, \{0\} \times \{0\})$ such that

$$\Psi \circ F' \circ \Phi = \alpha^*(F''),$$

where $\alpha^*(F'')$ is the unfolding $(x, \lambda) \mapsto (f''(x, \alpha(\lambda)), \lambda)$. This equality can be rewritten

$$(1 \times \alpha) \circ \Psi \circ F' \circ \Phi \circ (1 \times \alpha^{-1}) = F'',$$

and therefore to conclude that $\{F', g\}$ and $\{F'', g\}$ are $\mathcal{A}$-equivalent, it remains only to show that we can find a diffeomorphism θ such that

$$(1 \times \alpha) \circ \Psi \circ g \circ \theta = g.$$

In fact we construct θ^{-1}. Since

$$(1 \times \alpha) \circ \Psi \circ g(y, v) = (\psi(y, \sum v_j^2), \alpha(\sum v_j^2)),$$

we look for a diffeomorphism $\beta : (\mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$ such that $\alpha(\sum v_j^2) = \sum (\beta_j(v_1, \ldots, v_k))^2$. This seems easiest to do by working directly with power series; for example, when $k = 2$, and assuming for ease of notation that $\alpha'(0) = 1$, we can take

$$\beta(v_1.v_2) = (v_1(1+\alpha_2(v_1^2+2v_2^2)+\alpha_3(v_1^4+3v_1^2v_2^2+3v_2^4)+\cdots)^{1/2}, v_2(1+\alpha_2 v_2^2+\alpha_3 v_2^4+\cdots)^{1/2}),$$

where the α_i are the coefficients of the Taylor series of α. Now we find that

$$(\psi(y, \sum v_j^2), \alpha(\sum v_j^2)) = g(\psi(y, \sum v_j^2), \beta(v));$$

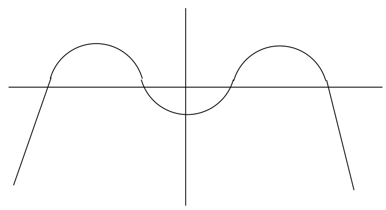
the right hand side of this equality is the composite of g with a diffeomorphism of its domain, and so we are done. □

When f has $\mathcal{A}_e$-codimension 1, the germ $g^*(f)$ obtained by applying the procedure of theorem 3.1 is the k-fold augmentation of f, $A^k f$. It will be useful to have a notation for the multi-germ $\{F, g\}$: we will denote it by $C_k(f)$. Both $A^k f$ and $C_k(f)$ are well-defined as $\mathcal{A}$-equivalence classes, by 2.1 and 3.3.

Example 3.4 Let $f = \{f_1, f_2, f_3, f_4\}$ be the stable multi-germ parametrising the union of the four coordinate hyperplanes $\{x_i = 0\}$ in $\mathbb{C}^4$ (in descending order of i), and let $g(x, y, z) = (x, y, z, z + y + x^k)$. Then by successive de-concatenation, the codimension and image Milnor number of the 5-germ $\{f, g\}$ are equal to those of the 4-germ $g^*(f)$ and the 3-germ $(g^*(f_1))^*(\{g^*(f_2), g^*(f_3), g^*(f_4)\})$. The latter is equivalent to

$$\begin{cases} x \mapsto (x, -x^k)) \\ x \mapsto (x, 0) \\ x \mapsto (0, x) \end{cases}.$$

This has $\mathcal{A}_e$-codimension and image Milnor number equal to k — an r-branch parametrised curve-germ in the plane has $\mu_I = \delta - r + 1$. It also has a god real perturbation, shown here when $k = 4$.



Example 3.5

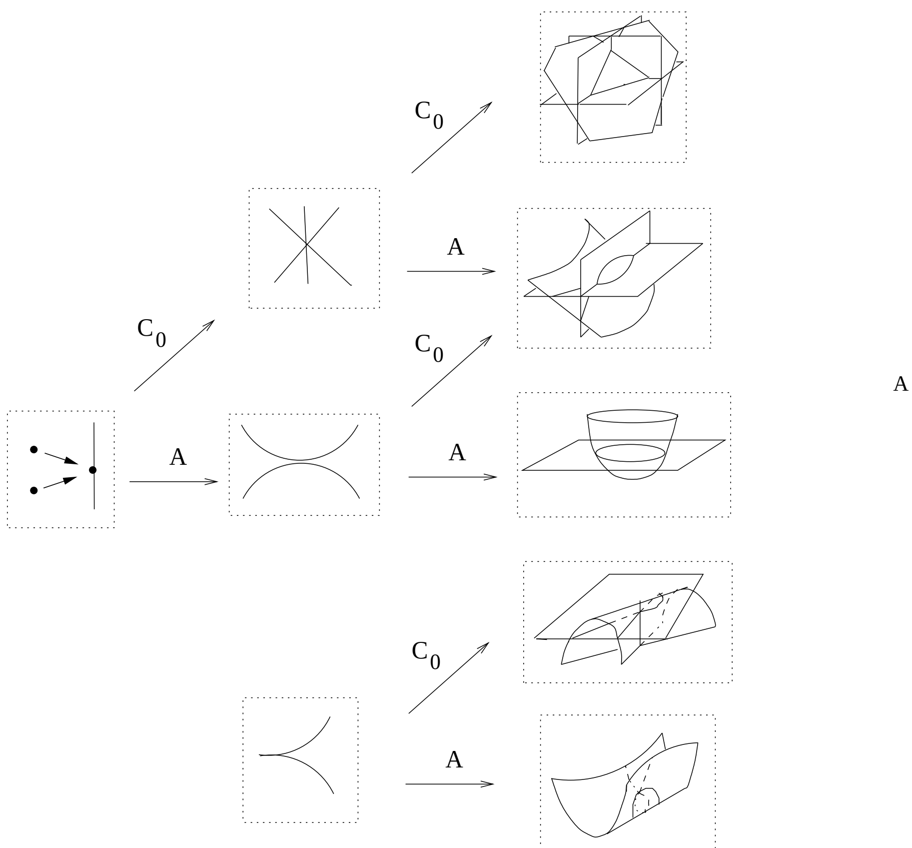


Figure 1: via A =Augmentation and C_0 =Concatenation, the double-point and the cusp generate all the codimension 1 map-germs from 2-space to 3-space

Example 3.6

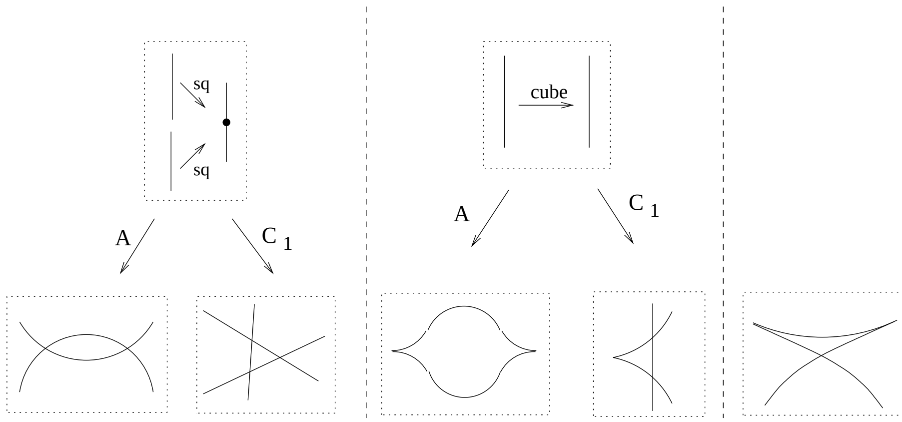


Figure 2: generation of codimension 1 germs of maps from the plane to the plane

Example 3.7 The bi-germ consisting of a cross cap together with an immersed plane transverse to the parametrisation of the cross-cap, and making contact of degree k with the double line in the cross-cap (cf 7.5 in [18], 3.3 in [26]) is obtained by applying C_0 to the germ $t \mapsto (t^2, t^{2k+1})$ parametrising the k-th order cusp.

The second type of concatenation is a binary operation: given germs $f_0 : (\mathbb{C}^m, S) \to (\mathbb{C}^a, 0)$ and $g_0 : (\mathbb{C}^n, T) \to (\mathbb{C}^b, 0)$ with 1-parameter stable unfoldings F and G, we form the multi-germ h essentially by putting together $\mathrm{id}_{\mathbb{C}^a} \times F$ and $G \times \mathrm{id}_{\mathbb{C}^b}$ so that their analytic strata (see Section 5) meet subtransversely in $\mathbb{C}^{a+b+1}$.

Theorem 3.8 *Suppose the two map-germs $F(y, s) = (f_s(y), s)$ and $G(x, s) = (g_s(x), s)$ are stable, and let h be defined by*

$$\begin{cases} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, u) \mapsto (g_u(x), Y, u) \end{cases}.$$

Then provided $\mathcal{A}_e$-codim $(h) < \infty$, we have

1. $$\mathcal{A}_e\text{-codim } (h) \geq \mathcal{A}_e\text{-codim } (f_0) \times \mathcal{A}_e\text{-codim } (g_0),$$
   with equality if and only if either $s \in ds(Der(\log D(G)))$ or $t \in dt(Der(\log D(F)))$;
2. *h has a 1-parameter stable unfolding;*
3. $$\mu_\Delta(h) = \mu_\Delta(f_0) \times \mu_\Delta(g_0).$$

Proof (1) and (2): we compute the codimension of h by Damon's theorem. The multi-germ

$$H : \begin{cases} (X, s, y, t) \mapsto (X, s, f_t(y), t) \\ (x, s, Y, t) \mapsto (g_s(x), s, Y, t) \end{cases}$$

is stable, as $\tau(F) \pitchfork \tau(G)$, and after a change of coordinates can be seen as an unfolding of h (which proves (2)). Our map h is induced from H by

$$i : \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C} \to \mathbb{C}^a \times \mathbb{C} \times \mathbb{C}^b \times \mathbb{C},$$

$$i(X, Y, u) = (X, u, Y, u).$$

The discriminant of H is the "product-union" (Jim Damon's term)

$$(D(G) \times \mathbb{C}^b \times \mathbb{C}) \bigcup (\mathbb{C}^a \times \mathbb{C} \times D(F)),$$

so if $\xi_0, \ldots, \xi_b$ generate $\mathrm{Der}(\log D(F))$ and $\eta_0, \ldots, \eta_a$ generate $\mathrm{Der}(\log D(G))$ then (considering the ξ_i as belonging to $\theta(a+1+b+1/a+1)$ and the η_i as belonging to $\theta(a+1+b+1/b+1)$) we have

$$N\mathcal{K}_{D(H),e}i = \theta(i)/\langle \partial/\partial X_i, \partial/\partial Y_j, \partial/\partial s + \partial/\partial t\rangle + \langle \xi_0, \ldots, \xi_b, \eta_0, \ldots, \eta_a\rangle.$$

Denote $ds(\mathrm{Der}(\log D(G)))$ and $dt(\mathrm{Der}(\log D(F)))$ by I and J respectively. By the map (ds, dt), $N\mathcal{K}_{D(H),e}i$ projects isomorphically to $M :=$

$$\frac{\mathcal{O}_{a+b+1}\langle \partial/\partial s, \partial/\partial t\rangle}{\langle \partial/\partial s + \partial/\partial t\rangle + \langle \{\alpha(X, u)\partial/\partial s : \alpha(X, s) \in I\}\rangle + \langle \{\beta(Y, u)\partial/\partial t : \beta(Y, t) \in J\}\rangle}.$$

As f_0 is induced from F by $\gamma(y) = (y, 0)$, and g_0 is induced from G by $\sigma(x) = (x, 0)$,

$$N\mathcal{A}_e f_0 \simeq \theta(\gamma)/t\gamma(\theta_b) + \gamma^*(\mathrm{Der}(\log D(F))) \overset{dt}{\simeq} \mathcal{O}_b/\gamma^*(J)$$

and

$$N\mathcal{A}_e g_0 \simeq \theta(\sigma)/t\sigma(\theta_a) + \sigma^*(\mathrm{Der}(\log D(G))) \overset{ds}{\simeq} \mathcal{O}_a/\sigma^*(I).$$

Now, suppose that $s \in I$. Then M is isomorphic to $M_0 :=$

$$\frac{\mathcal{O}_{a+b}\langle \partial/\partial s, \partial/\partial t\rangle}{\langle \partial/\partial s + \partial/\partial t\rangle + \mathcal{O}_{a+b}\sigma^*(I)\partial/\partial s \ + \ \mathcal{O}_{a+b}\gamma^*(J)\partial/\partial t}$$

The reason that $M \simeq M_0$ is that $u\partial/\partial s \in \{\alpha(X, u)\partial/\partial s) : \alpha(X, t) \in ds(\mathrm{Der}(\log D(G)))\}$ is in the denominator, and thus (since $\partial/\partial s + \partial/\partial t$ is in the denominator), so is $u\partial/\partial t$.

Evidently, if $t \in J$ then $M \simeq M_0$, by the same argument. An easy argument shows that the converse is true: if $M \simeq M_0$ then either $s \in I$ or $t \in J$.

The module M_0 is itself isomorphic to

$$\frac{\mathcal{O}_{a+b}}{\sigma^*(I)+\gamma^*(J)}$$

via the map $ds - dt$

$$\alpha\partial/\partial s + \beta\partial/\partial t \mapsto \alpha - \beta,$$

and finally, provided the left hand side is finite-dimensional,

$$\frac{\mathcal{O}_{a+b}}{\sigma^*(I)+\gamma^*(J)} \simeq \frac{\mathcal{O}_a}{\sigma^*(I)} \otimes_{\mathbb{C}} \frac{\mathcal{O}_b}{\gamma^*(J)}.$$

This completes the proof of (1).
(3) We postpone proof of this until Section 6. □

Remark 3.9 Let $f_0 : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a germ with a 1-parameter stable unfolding F, and suppose $n \geq p$ and (n,p) are nice dimensions. Then the condition in the proposition, that $t \in dt(\mathrm{Der}(\log D(F)))$, is equivalent to having $\mu_\Delta(f_0) = \mathcal{A}_e - \mathrm{codim}(f_0)$ - see [22], Corollary 7.4. The proof uses coherence of the Gauss-Manin connection.

Now suppose both f_0 and g_0 have $\mathcal{A}_e$-codimension 1. By analogy with augmentation and the first type of concatenation, one would expect the result of this second type of concatenation to be independent, up to $\mathcal{A}$-equivalence, of the choice of stable unfoldings F and G. Somewhat surprisingly, this is true over $\mathbb{C}$ but false over $\mathbb{R}$.

Example 3.10 Let $f_0(y) = y^3,\ g_0(x) = x^3$, and take $F'(y,u) = (y^3+uy,u),\ F''(y,u) = (y^3 - yu, u), G(x,u) = (x^3+ux,u)$. Then the multi-germs

$$h' : \begin{cases} (X,y,u) \mapsto (X, y^3+uy, u) \\ (x,Y,u) \mapsto (x^3+ux, Y, u) \end{cases}$$

and

$$h'' : \begin{cases} (X,y,u) \mapsto (X, y^3-uy, u) \\ (x,Y,u) \mapsto (x^3+ux, Y, u) \end{cases}$$

are not equivalent over $\mathbb{R}$. The discriminant of h' consists of two components, each the product of a first-order cusp with a line, and both "opening downwards" (in the direction of the negative u axis). This germ h' does not have a good real perturbation. On the other hand, in the germ h'' one cusp opens upwards and the other downwards, and h'' does have a good real perturbation, shown in Figure 3.

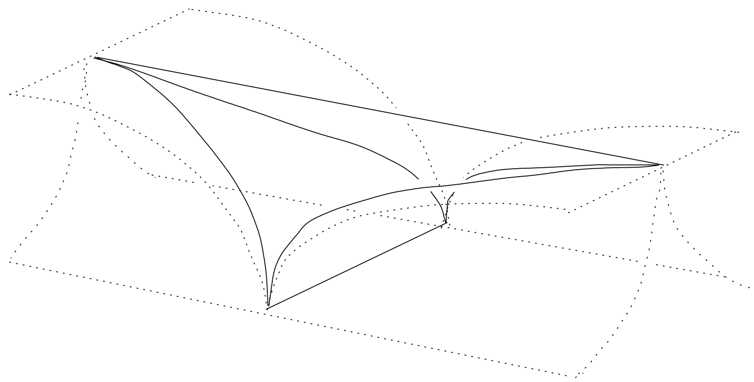


Figure 3: Discriminant of a good real perturbation of a binary concatenation of two cubic functions

Proposition 3.11 *Suppose that the germs f_0 and g_0 in Theorem 3.8 both have $\mathcal{A}_e$-codimension 1. Then over $\mathbb{C}$, and up to $\mathcal{A}$-equivalence, the germ h is independent of choice of the 1-parameter stable unfoldings F and G.*

Proof Suppose that F' and F'' are 1-parameter stable unfoldings of f_0, and let G be a 1-parameter stable unfolding of g_0. Applying the concatenation operation, we obtain multi-germs h' and h'', the first using F' and G, the second F'' and G. We wish to show that the two are $\mathcal{A}$- equivalent. Let h_λ be the linear interpolation between them: $h_\lambda = (1 - \lambda h') + \lambda h''$. We use a Mather-Yau type argument.

Step 1 For no value of λ is the germ h_λ stable.
For the analytic strata of its branches $\mathrm{id}_{\mathbb{C}^n} \times F_\lambda$ and G always meet at $0 \in \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$, and always have dimensions whose sum is less than $a + b + 1$, unless for some value of λ F_λ is a trivial unfolding of f_0. In the latter case F_λ itself is not stable, so that once again h_λ cannot be stable. It also follows that for those λ such that h_λ has $\mathcal{A}_e$-codimension 1, $T\mathcal{A}\, h_\lambda = T\mathcal{K}\, h_\lambda$.

Step 2 The set of points $\{\lambda \in \mathbb{C} : \mathcal{A}_e\text{−codimension}(h_\lambda) > 1\}$ is Zariski-closed in $\mathbb{C}$, so that its complement, $\Lambda_1 := \{\lambda \in \mathbb{C} : \mathcal{A}_e\text{−codimension}(h_\lambda) = 1\}$, is Zariski-open, and connected.

Step 3 Choose an integer k such that in the appropriate multi-jet space ${}_rJ^k(X, Y)$, the $J^k\mathcal{A}$-orbit of the k-jet of every codimension 1 germ coincides with the set of k-jets of its $\mathcal{A}$-orbit. We use Mather's Lemma ([16], 3.1) to show that the set $J^k\Lambda_1 := \{j^k h_\lambda : \lambda \in \Lambda_1\}$ lies in a single $J^k\mathcal{A}$-orbit, from which the proposition follows. It is necessary to check only that $T_\sigma J^k\Lambda_1 \subset TJ^k\mathcal{A}\sigma$ for all $\sigma \in \Lambda_1$. But $J^k\Lambda_1$ lies in a single contact orbit, and for each $\lambda \in \Lambda_1$, the $\mathcal{A}$-tangent space of h_λ is equal to its contact tangent space. It follows that $T_\sigma J^k\Lambda_1 \subset TJ^k\mathcal{A}\sigma$ for all $\sigma \in \Lambda_1$, as required. □

The argument of this proof in fact proves the following result, which we will use later:

Lemma 3.12 *In any given (complex) contact class there is at most one open $\mathcal{A}$-orbit.*

□

In the light of 3.11, we will refer to the $\mathcal{A}$-equivalence class of multi-germ obtained from codimension 1 multi-germs f_0 and g_0 by this binary concatenation operation as $B(f_{,0}, g_0)$.

Question How many different $\mathcal{A}$-equivalence classes of germs h over $\mathbb{R}$ can the different choices of miniversal unfoldings F, G of f_0 and g_0 give rise to?

Our final result here is

Proposition 3.13 *If the germs f_0 and g_0 both have good real perturbations, then so does $B(f_0, g_0)$.*

The proof will be given in Section 6.

Remark 3.14 It would be interesting to understand the effect on monodromy groups of augmentation and concatenation. There is a "natural" choice of 1-parameter stable unfolding of Af_0, $C_k(f_0)$ and of $B(f_0, g_0)$, reflecting the choice of stable unfolding used in their construction. Presumably the monodromy action in the case of $B(f_0, g_0)$ is the tensor product of the monodromy action in the chosen 1-parameter unfoldings F and G, as in the classical Thom-Sebastiani theorem.

4 $\mathcal{A}_e$-codimension 1 germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$

In this section we first classify $\mathcal{A}_e$-codimension 1 mono-germs and then show that each has image Milnor number 1. The argument runs roughly as follows: let

$$D^k(f) = \text{closure}\{(x_1, \dots, x_k) \in (\mathbb{C}^n, S)^k | x_i \neq x_j \text{ for } i \neq j, f(x_i) = f(x_j) \forall i, j\};$$

then by results of [13], f is stable if and only if $D^k(f)$, is smooth of dimension $n-k+1$, for $2 \leq k \leq n+1$,and f has finite $\mathcal{A}_e$-codimension if and only if each $D^k(f)$ is an isolated complete intersection singularity of dimension $n-k+1$, again for $0 \leq k \leq n+1$. Moreover, if f_t is a stable perturbation of f, then $D^k(f_t)$ is a Milnor fibre of $D^k(f)$. There is an obvious symmetric group action on $D^k(f)$, permuting the copies of $(\mathbb{C}^n, S)$, and in fact a spectral sequence ([7]) computes the homology of the imageof f_t from the S_k-alternating part of the homology of $D^k(f_t)$.It turns out that if f has $\mathcal{A}_e$-codimension 1, then just one of the $D^k(f)$ is singular, and in fact has a Morse singularity. Since the symmetric group action on the Jacobian algebra is therefore trivial, from a theorem of Orlik-Solomon and Wall it follows that the vanishing homology of $D^k(f_t)$ is alternating, and thus by the spectral sequence the image Milnor number is 1. The symmetry of $D^k(f)$ also accounts for the existence of a good real perturbation. Essentially, the point is that an S_k-invariant Morse function in k real variables is either a sum of squares or the negative of a sum of squares.

Now we proceed with the classification. Let $\ell > 0$, take coordinates $(u_1, \cdots, u_{\ell-1}, v_1, \cdots, v_{\ell-1}, x)$ on $\mathbb{C}^{2\ell-1}$, and define a map
$f^\ell : (\mathbb{C}^{2\ell-1}, 0) \to (\mathbb{C}^{2\ell}, 0)$ by

$$f^\ell(u, v, x) = (u, v, x^{\ell+1} + \sum_{i=1}^{\ell-1} u_i x^i, x^{\ell+2} + \sum_{i=1}^{\ell-1} v_i x^i).$$

Lemma 4.1 *The map-germ f^ℓ just described has $\mathcal{A}_e$-codimension 1, and the following property:*
$(*)$: $D^k(f^\ell)$ is smooth for $2 \leq k \leq \ell$, $D^{\ell+1}(f^\ell)$ has a Morse singularity, and $D^k(f^\ell)$ is empty for $k > \ell + 1$.

Proof Recall from [13] 2.1 the determinantal equations $h^k_{j,i}$ of $D^k(f^\ell)$:

$$h^k_{j,i} = \frac{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{i-1} & f_j^\ell(u,v,x_1) & x_1^{i+1} & \cdots & x_1^{k-1} \\ & & & & \cdot & & & \\ & & & & \cdot & & & \\ & & & & \cdot & & & \\ 1 & x_k & \cdots & x_k^{i-1} & f_j^\ell(u,v,x_k) & x_1^{i+1} & \cdots & x_k^{k-1} \end{vmatrix}}{\text{vdM}}$$

for $1 \leq i \leq k-1$ and $2\ell - 1 \leq j \leq 2\ell$, where vdM is the van der Monde determinant of $x_1, \cdots, x_k$, and f_j^ℓ is the j'th component of f^ℓ. An easy calculation shows

$$h^k_{2\ell-1,i} = u_i + O(2) \qquad \text{for } i = 2, \cdots, \ell - 1$$

$$h^k_{2\ell,i} = v_i + O(2) \qquad \text{for } i = 2, \cdots, \ell - 1$$

so that $D^k(f^\ell)$ is smooth for $2 \leq k \leq \ell$; moreover

$$h^{\ell+1}_{2\ell-1,\ell} = x_1 + \cdots + x_{\ell+1}$$

and

$$h^{\ell+1}_{2\ell,\ell} = \sum_{i,j=1}^{\ell+1} x_i x_j.$$

We may take the $h^{\ell+1}_{j,i}$ for $2 \leq i \leq \ell - 1$, together with $x_1, \cdots, x_{\ell+1}$, as coordinates; then $D^{\ell+1}(f)$ is embedded in $x_1, \cdots, x_{\ell+1}$-space with equations $h^{\ell+1}_{2\ell-1,\ell}$ and $h^{\ell+1}_{2\ell,\ell}$. Now $h^{\ell+1}_{2\ell-1,\ell}$ is non-singular, and

$$h^{\ell+1}_{2\ell,\ell} - \frac{1}{2}(h^{\ell+1}_{2\ell-1,\ell})^2 = \sum_{i=1}^{\ell+1} x_i^2,$$

so $D^{\ell+1}(f^\ell)$ has a Morse singularity at the origin.

Calculation of the $\mathcal{A}_e$-codimension of f^ℓ is straightforward; it may easily be checked using nothing more than Nakayama's Lemma that

$$T\mathcal{A}_e f^\ell = \theta(f) \setminus \{x^\ell \partial/\partial Y_2, x^{\ell-1}\partial/\partial v_1, \cdots, x\partial/\partial v_{\ell-1}\} +$$

$$+\langle x^{\ell-1}\partial/\partial v_1 + x^\ell \partial/\partial Y_2, \cdots, x\partial/\partial v_{\ell-1} + x^\ell \partial/\partial Y_2\rangle.$$

The calculation is carried out in detail in [2]. □

Note that since f^ℓ has $\mathcal{A}_e$-codimension 1, its $\mathcal{A}$-orbit is open in its $\mathcal{K}$-orbit. Note also that from the expression for $T\mathcal{A}_e f^\ell$ given in the proof, it follows that the stable germ

$$F(\lambda, u, v, x) = (\lambda, u, v, x^{\ell+1} + \sum_{i=1}^{\ell-1} u_i x^i, x^{\ell+2} + \sum_{i=1}^{\ell-1} v_i x^i + \lambda x^\ell)$$

is an $\mathcal{A}_e$-versal unfolding of f^ℓ.

Since for corank 1 germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ the multiplicity determines the contact class, it follows from Lemma 3.12 that we have

Corollary 4.2 *If $f : (\mathbb{C}^{2\ell-1}, 0) \to (\mathbb{C}^{2\ell}, 0)$ has corank 1, multiplicity $\ell+1$ and $\mathcal{A}_e$-codimension 1, then f is $\mathcal{A}$-equivalent to the germ f^ℓ of Lemma 4.1.* □

Proposition 4.3 *If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ has corank 1, multiplicity $\ell + 1$ and $\mathcal{A}_e$-codimension 1 then it is equivalent to*

$$f_q^\ell : (u, v, w, x) \mapsto (u, v, w, x^{\ell+1} + \sum_{i=1}^{\ell-1} u_i x^i, x^{\ell+2} + \sum_{i=1}^{\ell-1} v_i x^i + q(w) x^\ell)$$

where q is a non-degenerate quadratic form.

Proof Note that f_q^ℓ is (over $\mathbb{C}$) equivalent to the k-fold augmentation $A^k f^\ell$, where $k = n - 2\ell$. The hypothesis forces $n \geq 2\ell - 1$, since the minimal target dimension of a stable corank 1 germ of multiplicity $\ell+1$ is $2\ell+1$. Since f has $\mathcal{A}_e$-codimension 1, its versal unfolding $G : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}, 0)$ is an $n - 2\ell + 1$-fold prism on a minimal stable map-germ of multiplicity $\ell + 1$. From this it follows by Theorem 2.7 that f is equivalent to an $n - 2\ell + 1$-fold augmentation of an $\mathcal{A}_e$-codimension 1 germ $f_0 : (\mathbb{C}^{2\ell-1}, 0) \to (\mathbb{C}^{2\ell}, 0)$ of multiplicity $\ell + 1$ and corank 1. By the previous corollary, f_0 is equivalent to the germ f^ℓ of 4.1; since the germ F described after 4.1 is a versal unfolding of f^ℓ, f is equivalent to the germ obtained by replacing the unfolding term λx^ℓ in the last component of F by $q(w)x^\ell$, where q is a non-degenerate quadratic form in new variables w_i, as required. □

Proposition 4.4 *If $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ has corank 1 and $\mathcal{A}_e$-codimension 1 then $\mu_I(f) = 1$, and there is a real form with a good real perturbation.*

Proof Let f_t be a stable perturbation of f, with image Y_t. By [7] Theorem 2.5,

$$H^n(Y_t;\mathbb{Q}) \simeq \oplus_k \mathrm{Alt}_k H^{n-k+1}(D^k(f_t);\mathbb{Q}) \tag{1}$$

where $\mathrm{Alt}_k H^{n-k+1}(D^k(f_t);\mathbb{Q})$ means the subspace of $H^{n-k+1}(D^k(f_t);\mathbb{Q})$ on which the symmetric group S_k acts by its sign representation. Now $D^k(f_t)$ is a Milnor fibre of $D^k(f)$; since f has property $(*)$, (1) reduces to

$$H^n(Y_t;\mathbb{Q}) \simeq \mathrm{Alt}_{\ell+1} H^{n-\ell}(D^{\ell+1}(f_t);\mathbb{Q}).$$

As $D^{\ell+1}(f)$ has a Morse singularity, $H^{n-\ell}(D^{\ell+1}(f_t);\mathbb{Q}) \simeq \mathbb{Q}$; it remains to show that the representation of $S_{\ell+1}$ on $H^{n-\ell}(D^{\ell+1}(f_t);\mathbb{Q})$ is the sign representation. This can easily be seen by an explicit calculation with the normal form given; but there is another argument which explains better why it is true. As $D^{\ell+1}(f)$ is an $S_{\ell+1}$-invariant hypersurface singularity, by the theorem of Orlik-Solomon and Wall ([23],[25]),

$$H^{n-\ell}(D^{\ell+1}(f_t);\mathbb{Q}) \simeq \mathrm{Jac}_{D^{\ell+1}(f)} \otimes_{\mathbb{Q}} \wedge^{\ell}(V)^*$$

as $S_{\ell+1}$ representations, where V is an $S_{\ell+1}$-invariant smooth space containing $D^{\ell+1}(f)$ as a hypersurface, and $\mathrm{Jac}_{D^{\ell+1}(f)}$ is the Jacobian algebra of $D^{\ell+1}(f)$. Since $D^{\ell+1}(f)$ is Morse, its Jacobian algebra is a trivial 1-dimensional representation of $S_{\ell+1}$, so $H^{n-\ell}(D^{\ell+1}(f_t);\mathbb{Q}) \simeq \wedge^{\ell}(V)^*$. In fact we take $V = D^{\ell+1}(G)$ where G is a 1-parameter stable unfolding of f; as noted above, G is right-left equivalent to a suspension of F, and in particular the $S_{\ell+1}$-action on $D^{\ell+1}(G)$ is equivalent to a trivial extension of the standard Weyl action A_ℓ, in which $S_{\ell+1}$ acts on $\{(x_1,\cdots,x_{\ell+1}) : \sum_i x_i = 0\}$ by permuting coordinates. Hence $\wedge^{\ell}(V)^*$ is just the sign representation of $S_{\ell+1}$ and (as vector spaces)

$$H^{2\ell-1}(Y_t;\mathbb{Q}) \simeq \mathrm{Alt}_{\ell+1} H^{n-\ell}(D^{\ell+1}(f_t);\mathbb{Q}) = \mathbb{Q}$$

so that $\mu_I(f) = 1$.

In the real case, we apply (1) to a real stable perturbation $f_{t,\mathbb{R}}$ of f, replacing $D^k(f_t)$ by $D^k(f_{\mathbb{R},t})$. Consider first the case $n = 2\ell-1$, so f is equivalent to the germ f^ℓ of 4.1. Let $f^C_{\mathbb{R},t}$ be a stable perturbation. Evidently $D^k(f^C_{\mathbb{R},t})$ is contractible for $2 \le k < \ell+1$, and $D^{\ell+1}(f_{\mathbb{R},t})$ is a real Milnor fibre of a $\ell-1$-dimensional Morse singularity; hence it is a p-sphere for some p between -1 and $\ell-1$. We have to show that either for $t>0$ or $t<0$ it is an $\ell-1$-sphere. This follows from the fact that $D^{\ell+1}(f^\ell)$ has a Morse singularity and an $S_{\ell+1}$-invariant defining equation, in a space in which the representation of $S_{\ell+1}$ is equivalent to the Weyl representation A_ℓ described above. Since the representation is irreducible, the stable manifold and unstable manifold of the gradient flow must be equal to 0 and V or V and 0 respectively, and any $S_{\ell+1}$-invariant quadratic form must have index 0 or ℓ. Since the versal unfolding F of f^ℓ is a stable map, $D^{\ell+1}(F)$ is smooth, and thus projection to the parameter space cuts out distinct real Milnor fibres for $t>0$ and $t<0$. Hence at least one of these is an ℓ-sphere. Inclusion $D^{\ell+1}(f^\ell_{\mathbb{R},t}) \hookrightarrow D^{\ell+1}(f^\ell_t)$ then induces an $S_{\ell+1}$-equivariant

homotopy equivalence, so that the representation of $S_{\ell+1}$ on $H^{\ell-1}(D^{\ell+1}(f^{\ell}_{\mathbb{R},t}))$ is once again the sign representation.

In the general case, let $f^{\ell}_{q\ \mathbb{R},t}$ be a stable perturbation of f^{ℓ}_q. By taking $q = \sum_i w_i^2$, then $D^{\ell+1}(f_t)$ is an $\ell - 1 + d$-dimensional sphere, where $d = n - 2\ell + 1$ is the number of w-variables in the expression for f^{ℓ}_q in 4.3. In fact $D^{\ell+1}(f^{\ell}_{q\ \mathbb{R},t})$ is the join of $D^{\ell+1}(_{\mathbb{R},t})$ and $q^{-1}(t)$, and the representation of $S_{\ell+1}$ on its cohomology is just the sign representation as before. □

Remark 4.5 The argument just used shows that if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ has corank 1 and multiplicity $\ell + 1$, and has a 1-parameter stable unfolding F, and if $D^k(f)$ is singular, then $\mu_I(f) \geq \ell + 2 - k$. For from the fact that $D^k(f)$ is singular it follows that $D^j(f)$ is singular, for $k \leq j \leq \ell + 1$. As $D^j(f)$ is a hypersurface in the smooth space $D^j(F)$, the argument used above can be applied. The Jacobian algebra of each singular $D^k(f)$ has S_k-invariant subspace of dimension at least 1 (since the constants form a 1-dimensional trivial representation), and hence by the theorem of Wall (rather than the earlier result of Orlik-Solomon, which applies only to weighted homogeneneous hypersurface singularities) the alternating part of the middle homology of the Milnor fibre $D^k(f_t)$ has rank at least 1. The conclusion then follows by (1).

5 $\mathcal{A}_e$-codimension 1 multi-germs

In this section we show that in the nice dimensions all $\mathcal{A}_e$-codimension 1 multi-germs can be constructed by concatenation and augmentation, beginning with stable germs and with primitive $\mathcal{A}_e$-codimension 1 mono-germs.

Submersive branches of multi-germs play a trivial role in classification and deformation theory, and we will ignore them in what follows. In particular "a multi-germ with k branches" means a multi-germ with k non-submersive branches.

For a multi-germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with branches $f^{(1)}, \ldots, f^{(s)}$, define

$$\tau(f) = ev_0[(\omega f)^{-1}\{f^* m_p \theta(f) + tf(\theta(n)_S)\}]$$

where $ev_0 : \theta(p) \to T_0\mathbb{C}^p$ is evaluation at 0, and

$$\tau'(f) = ev_0[(\omega f)^{-1}\{tf(\theta(n)_S)\}]$$

In fact $\tau'(f) = ev_0(\mathrm{Der}(\log D(f)))$ where $D(f)$ is the discriminant (or image) of f.

The following result is due to Mather [16].

Proposition 5.1 *The multi-germ f is stable if and only if each $f^{(i)}$ is stable and $\tau(f^{(1)})$, $\ldots, \tau(f^{(s)})$ have regular intersection with respect to $T_0\mathbb{C}^p$. Moreover, in this case $\tau(f) = \bigcap_i \tau(f^{(i)})$.* □

We now investigate the geometrical significance of τ'.

Lemma 5.2 *If* $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ *is stable then* $\tau(f)=\tau'(f)$. $\square$

Lemma 5.3 *If* $f=id_{\mathbb{C}^m}\times g$ *(i.e.* $f=P^m g$*) then* $\tau'(f)=T_0\mathbb{C}^m\oplus\tau'(g)$. $\square$

Lemma 5.4 *If* $dim_{\mathbb{C}}\tau'(f)=m$*, then there is a germ* g*, not a prism, such that* $f\sim_{\mathcal{A}}P^m g$*. Moreover, if* ϕ *and* ψ *are diffeomorphisms such that* $f\circ\phi=\psi\circ(id_{\mathbb{C}^m}\times g)$*, then* $\tau'(f)=d\psi_0(T_0\mathbb{C}^m\times\{0\})$.

Proof Suppose $tf(\xi)=\omega f(\eta)$. If $\eta(0)\neq 0$ then also $\xi(s)\neq 0$ for $s\in S$, and the orbits of ξ and η can be incorporated as coordinate lines into new coordinate systems on $\mathbb{C}^n,S$ and $\mathbb{C}^p,0$; now the lemma just reduces to the Thom-Levine Lemma (see e.g.[24]), and $f\sim_{\mathcal{A}}Pg_1$ for some germ g_1. Now apply the same procedure to g_1. After m iterations, we arrive eventually at a g with $\tau'(g)=0$, which is therefore not a prism. $\square$

Proposition 5.5 *If* $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ *and* $g:(\mathbb{C}^m,T)\to(\mathbb{C}^q,0)$ *are multi-germs neither of which are prisms and if* $P^k f$ *is* $\mathcal{A}$*-equivalent to* $P^\ell g$ *then* $|S|=|T|$, $n=m$, $p=q$, $k=\ell$ *and* f *is* $\mathcal{A}$*-equivalent to* g*. Furthermore, if the* $\mathcal{A}$*-equivalence between* $P^k f$ *and* $P^\ell g$ *is given by diffeomorphisms* ϕ *and* ψ *as in the following diagram then* $\psi(\mathbb{C}^k\times\{0\})=\mathbb{C}^\ell\times\{0\}$

$$\begin{array}{ccc}
\mathbb{C}^k\times\mathbb{C}^n,\{0\}\times S & \stackrel{id_{\mathbb{C}^k}\times f}{\rightarrow} & \mathbb{C}^k\times\mathbb{C}^p,(0,0)\\
\downarrow\phi & & \downarrow\psi\\
\mathbb{C}^\ell\times\mathbb{C}^m,\{0\}\times T & \stackrel{id_{\mathbb{C}^\ell}\times g}{\rightarrow} & \mathbb{C}^\ell\times\mathbb{C}^q,(0,0)
\end{array}$$ $\square$

Given a multi-germ f, by Proposition 5.5 there is a well defined maximal sub-manifold of the target along which f is trivial (i.e. a prism). It is known as the **analytic stratum** of f, and coincides with the set-germ of points $y\in\mathbb{C}^p,0$ such that the germ $f:(\mathbb{C}^n,f^{-1}(y)\cap C_f)\to(\mathbb{C}^p,y)$ is $\mathcal{A}$-equivalent to $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$. Moreover, $\tau'(f)$ is the tangent space at 0 to the analytic stratum of f.

Proposition 5.6 *Let* $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ *and* $g:(\mathbb{C}^n,T)\to(\mathbb{C}^p,0)$ *be multi-germs, and suppose that* $h=\{f,g\}$ *has* $\mathcal{A}_e$*-codimension 1. Let* ψ *be a germ of a 1-parameter family of maps* $(\mathbb{C}^p,0)\to\mathbb{C}^p$ *such that* $\psi_0=id_{\mathbb{C}^p}$ *and*

$$ev_0(\frac{d\psi_t}{dt}|_{t=0})\notin\tau'(f)+\tau'(g),$$

and write $G(\lambda,x)=(\lambda,(\psi_\lambda\circ g)(x))$*. Then* $H:=\{id_{\mathbb{C}}\times f,G\}$ *is a versal unfolding of* h.

Proof Write $H(\lambda,x)=(\lambda,h_\lambda(x))$. If $v=\frac{dh_\lambda}{d\lambda}|_{\lambda=0}\in T\mathcal{A}_e h$, then $v=th(\xi)+\omega h(\eta)$ for some $\xi\in\theta(n)_{S\cup T}$ and $\eta\in\theta(p)$. It follows that $\omega f(\eta)=tf(-\xi)$ and $tg(-\xi)=\omega g(\eta-\frac{d\psi_t}{dt}|_{t=0})$ and therefore $ev_0(\frac{d\psi_t}{dt}|_{t=0})\in\tau'(f)+\tau'(g)$, which contradicts our hypotheses.

Since $\frac{dh_\lambda}{d\lambda}|_{\lambda=0}\notin T\mathcal{A}_e h$ and h has $\mathcal{A}_e$-codimension 1, H is a versal unfolding of h. $\square$

Corollary 5.7 *If $h : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is a multi-germ of $\mathcal{A}_e$-codimension 1, then for every proper subset S' of S, the restriction of h to a multi-germ $(\mathbb{C}^n, S') \to (\mathbb{C}^p, 0)$ is stable.*

Proof Let $S = S' \cup S''$ with $S' \cap S'' = \emptyset$. Let h' and h'' be the multi-germs of h at S' and S'' respectively. Suppose that one of h' and h'' is not stable, say h'. Then h' has $\mathcal{A}_e$-codimension 1. Since it is therefore not a prism, by Lemma 5.4 $\tau'(h') = 0$. As h'' is not a submersion, we may choose $v \in T_0\mathbb{C}^p \backslash \tau'(h'')$. Extend v to a vector field on $\mathbb{C}^p$ and integrate it to give a germ of a 1-parameter family ψ_t of diffeomorphisms of $(\mathbb{C}^p, 0)$ satisfying the conditions of Proposition 5.6. Therefore H, as described in Proposition 5.6, is a versal unfolding of h. But then $id_{\mathbb{C}} \times h'$ is a versal unfolding of h' and so h' is stable, a contradiction. Therefore h' and h'' are stable. $\square$

A finite set $E_1, \ldots, E_s$ of vector subspaces of a finite dimensional vector space F has **almost regular intersection** (with respect to F) if

$$\operatorname{codim}(E_1 \cap \ldots \cap E_s) = \operatorname{codim}E_1 + \cdots + \operatorname{codim}E_s - 1$$

Lemma 5.8 *$E_1, \ldots, E_s$ have almost regular intersection if and only if the cokernel of the natural mapping*

$$F \to (F/E_1) \oplus \ldots \oplus (F/E_s)$$

has dimension 1. $\square$

Proposition 5.9 *Let $h = \{f, g\}$ be an $\mathcal{A}_e$-codimension 1 multi-germ. Then $\tau(f)$ and $\tau(g)$ have almost regular intersection with respect to $T_0\mathbb{C}^p$.*

Proof Let H be a versal unfolding of h. H restricts to a versal unfolding F of f and a versal unfolding G of g. Since f is stable, F is equivalent to a prism on f and hence

$$T_0\mathbb{C}^p/\tau(f) \cong T_0(\mathbb{C} \times \mathbb{C}^p)/\tau(F)$$

We have the following commutative diagram

$$\begin{array}{ccc} T_0\mathbb{C}^p & \rightarrow & \frac{T_0\mathbb{C}^p}{\tau(f)} \oplus \frac{T_0\mathbb{C}^p}{\tau(g)} \\ \downarrow & & \downarrow \\ T_0(\mathbb{C} \times \mathbb{C}^p) & \rightarrow & \frac{T_0(\mathbb{C}\times\mathbb{C}^p)}{\tau(F)} \oplus \frac{T_0(\mathbb{C}\times\mathbb{C}^p)}{\tau(G)} \end{array}$$

in which the right hand map is bijective and the bottom map is surjective by 5.1. So the top map has cokernel of codimension at most 1. Were it surjective, then $\tau(f)$ and $\tau(g)$ would be transverse, and h would be stable. Hence the dimension of the cokernel is 1, proving the proposition. $\square$

Corollary 5.10 *If h is a multi-germ of $\mathcal{A}_e$-codimension 1 with branches $h^{(1)}, \ldots, h^{(r)}$, $r \geq 2$, then $\tau(h^{(1)}), \ldots, \tau(h^{(r)})$ have almost regular intersection with respect to $T_0\mathbb{C}^p$.* □

Corollary 5.11 *Let $h = \{f, g\}$ have $\mathcal{A}_e$-codimension 1. Then the codimension of $\tau(f) + \tau(g)$ in $T_0\mathbb{C}^p$ is 1.* □

It is natural to ask how we can tell when our codimension 1 multi-germ is primitive.

Proposition 5.12 *Let $h = \{f, g\}$ be an $\mathcal{A}_e$-codimension 1 multi-germ, and let $k = dim_{\mathbb{C}}$ $\tau(f) \cap \tau(g)$. Then h is a k-fold augmentation of a primitive map-germ.*

Proof By Corollary 5.10 we can choose $v \in T_0\mathbb{C}^p \setminus (\tau(f) + \tau(g))$. Choose a germ of a one parameter family ψ_t of diffeomorphisms of $(\mathbb{C}^p, 0)$ such that $ev_0(\frac{d\psi_t}{dt}|_{t=0}) = v$. Then choose a versal unfolding H of h as in Proposition 5.6. If Λ is the first coordinate in the target $\mathbb{C} \times \mathbb{C}^p$ of H then $\tau(F) = \mathbb{C}\frac{\partial}{\partial\Lambda} \oplus \tau(f)$ and $\tau(G) = \mathbb{C}(\frac{\partial}{\partial\Lambda} + v) \oplus \tau(g)$. Since $\tau(H) = \tau(F) \cap \tau(G)$ it follows that $\tau(H) = \tau(f) \cap \tau(g)$. Therefore, by Proposition 5.5, H is a prism and by Theorem 2.7 h is an augmentation. □

Corollary 5.13 *Suppose that $h = \{f, g\}$ is a primitive $\mathcal{A}_e$-codimension 1 multi-germ. Then there is a decomposition*

$$T_0\mathbb{C}^p = \tau(f) \oplus \tau(g) \oplus \mathbb{C}v.$$

Proof Immediate from Corollary 5.11 and Proposition 5.12. □

Example 5.14 Using 5.12 we classify codimension 1 multi-germs of immersions. If $f : \mathbb{C}^n, S \to \mathbb{C}^{n+1}$ has all of its r branches immersions, then the same is true of a 1-parameter versal unfolding F. As F is stable, these r branches meet in general position, with intersection L of dimension $n + 1 - r$. As $L = \tau(F)$, by 5.12 F is the $n + 1 - r$-fold augmentation of a germ $f_0 : \mathbb{C}^{r-1}, S \to \mathbb{C}^r, 0$, evidently also consisting of r immersions. As f_0 has $\mathcal{A}_e$-codimension 1, a little thought shows that each $r - 1$-tuple of its immersions is in general position (but see also 5.9). It follows that f_0 is equivalent to the germ consisting of a parametrisation of the r coordinate hyperplanes, together with one extra immersive branch $(x_1, \ldots, x_{r-1}) \mapsto (x_1, \ldots, x_{r-1}, \sum_i x_i)$. This has a versal unfolding in which only the last immersion is deformed, to $(x_1, \ldots, x_{r-1}) \mapsto (x_1 + t, \ldots, x_{r-1} + t, \sum_i x_i + t)$. Thus f is equivalent to the germ consisting of a parametrisation of the first $r - 1$ hyperplanes together with an additional immersion of the form

$$(x_1, \ldots, x_{r-1}, u_1, \ldots, u_{n+r-1}) \mapsto (x_1 + \sum_j u_j^2, \ldots, x_{r-1} + \sum_j u_j^2, \sum_i x_i + \sum_j u_j^2, u_1, \ldots, u_{n-r+1}).$$

In the real case, the only change in the classification is that $\sum_j u_j^2$ must be replaced by $\sum_j \pm u_j^2$, giving $(n-r+1)/2$ different classes if $n+r-1$ is even, or $(n+r)/2$ if $n-r+1$ is odd.

The second germ in the list shown in the right-hand column in Figure 1 on page 10 is of this type.

In view of 5.13, by a change of coordinates we can arrange that the analytic stratum of f becomes $\mathbb{C}^a \times \{0\} \times \{0\}$, that of g becomes $\{0\} \times \mathbb{C}^b \times \{0\}$ and v becomes $(0, 0, 1) \in \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$. We shall suppose for the remainder of this section that this change of coordinates has been made.

We say that a multi-germ f is transverse to a vector subspace V of $T_0\mathbb{C}^p$ if every branch of f is transverse to V. Our analysis of multi-germs $h = \{f, g\}$ from now on falls into two cases, characterised by whether g is or is not transverse to $\tau(f)$.

Case 1: g is not transverse to $\tau(f)$.

Lemma 5.15 *A stable map germ of rank zero is either a Morse singularity, or either the domain or the codomain has dimension zero.* □

Proposition 5.16 *Let $h = \{f, g\}$ be a primitive $\mathcal{A}_e$-codimension 1 multi-germ, and suppose that g is not transverse to $\tau(f)$. Then*

1. *if moreover g and f are transverse, it follows that*
   (a) *g has precisely one branch, which is either a prism on a Morse singularity or an immersion.*
   (b) *After a change of coordinates, h takes the form*

$$\begin{cases} f : (\mathbb{C}^{n-1} \times \mathbb{C}, S_0 \times \{0\}) \to (\mathbb{C}^{p-1} \times \mathbb{C}, 0), & f(x, u) = (f_u(x), u) \\ g : (\mathbb{C}^{p-1} \times \mathbb{C}^k, 0) \to (\mathbb{C}^{p-1} \times \mathbb{C}, 0), & g(\lambda, v) = (\lambda, \sum_j v_j^2) \end{cases}$$

   where f is an $\mathcal{A}_e$-versal unfolding of f_0; thus $h = C_k(f_0)$. In particular, $f \pitchfork \tau(g)$.

2. *if g and f are not transverse, then $p = 1$, and f and g are both Morse functions.*

Proof If g has more than one branch, then by 5.7 the multi-germ consisting of f together with any one branch $g^{(i)}$ of g is stable. Hence $\tau(g^{(i)}) \pitchfork \tau(f)$, so $g^{(i)} \pitchfork \tau(f)$, so $g \pitchfork \tau(f)$. This contradiction implies that g has only one branch.

Now suppose that Image$(dg(0))$ is bigger than $\tau(g)$. Then we can construct a 1-parameter deformation h_t of h by fixing f and composing g with a 1-parameter rotation about $\tau(g)$, in such a way that for $t \neq 0$, g becomes transverse to $\tau(f)$. Since $\tau(g)$ remains non-transverse to $\tau(f)$, h_t is not stable even for $t \neq 0$. But neither is it equivalent to $h = h_0$. This is impossible, since h has $\mathcal{A}$-codimension 1. Hence Image$(dg(0)) = \tau(g)$, and

so g is a prism on a germ of rank 0. By 5.15, g is either a prism on a Morse function or an immersion.
(2) The codimension of $\tau(g)$ is now 1, so by Corollary 5.13 we must have $\tau(f) = \{0\}$. Thus, we have a decomposition of the target as $\mathbb{C}^{p-1} \times \mathbb{C}$ where $\mathbb{C}^{p-1} \times \{0\}$ is the analytic stratum of g. There is a neighbourhood U of 0 in $\mathbb{C}^{p-1}$ such that for all $u \in U$, the pullback of g along the inclusion of the subset $\{u\} \times \mathbb{C}$ is a Morse singularity and so by a coordinate change in the source we can reduce this pullback to the form $\sum_{i=1}^{m} x_i^2$. In fact the changes of coordinates in the source depend analytically on u and so together they give a change of coordinates in the source which reduces g to the form

$$\begin{array}{ccc} \mathbb{C}^{p-1} \times \mathbb{C}^k & \to & \mathbb{C}^{p-1} \times \mathbb{C} \\ (\lambda, v_1, \ldots, v_k) & \mapsto & (\lambda, \sum_{j=1}^{m} v_j^2) \end{array}$$

Now suppose that f is transverse to g. Then by a change of coordinates in the source of f we can now bring f to the desired form. Evidently f is now a stable 1-parameter unfolding of f_0, so we can view h as $C_k(f_0)$; finally, by Theorem 3.1

$$\mathcal{A}_e - \mathrm{codim}(f_0) = \mathcal{A}_e - \mathrm{codim}(h) = 1.$$

On the other hand, if f is not transverse to g then we can apply the previous argument with the roles of f and g reversed, to conclude that $p = 1$ and thus that f and g are both Morse singularities. □

Example 5.17 The germ f_0 of Example 5.14 is obtained (up to $\mathcal{A}$-equivalence) by applying the concatenation operation C_0 (defined using Theorem 3.3) $r - 1$ times to the bi-germ consisting of coincident embeddings of two copies of $\mathbb{C}^0$ in $\mathbb{C}$.

To complete our analysis of codimension 1 multi-germs, by 5.16 it remains to consider only
Case 2: $f \pitchfork \tau(g)$ **and** $g \pitchfork \tau(f)$. Recall that we were able to decompose the target $\mathbb{C}^p$ as $\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$, with $\tau(f) = \mathbb{C}^a \times \{0\} \times \{0\}$ and $\tau(g) = \{0\} \times \mathbb{C}^b \times \{0\}$. Let $z_1, \ldots, z_{a+b+1}$ be coordinates on $\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$. Since f is transverse to $\tau(g)$, we can take $z_{a+b+1} \circ f$ as a coordinate, u, on the domain of f, and similarly, as g is transverse to $\tau(f)$, we can take $v = z_{a+b+1} \circ g$ as a coordinate on the domain of g. A coordinate change now brings $\{f, g\}$ to the form

$$\left\{ \begin{array}{l} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, u) \mapsto (g_{u,Y}(x), Y, u) \end{array} \right. .$$

Note that we have reduced f to a prism on a 1-parameter unfolding (using the fact that $\mathbb{C}^a \times \{0\} \times \{0\}$ is the analytic stratum of f), but that we have not done the same for g — yet. A naive coordinate change to reduce g to a prism on a 1-parameter unfolding would take f out of its normal form. Nevertheless, we claim that h is $\mathcal{A}$-equivalent to a binary concatenation of two $\mathcal{A}_e$-codimension 1 germs, as described in Section 3. As a first step, we prove:

Lemma 5.18 *Suppose that h is an $\mathcal{A}_e$-codimension 1 germ in the semi-normal form*

$$\begin{cases} (X,y,u) \mapsto (X, f_u(y), u) \\ (x,Y,u) \mapsto (g_{u,Y}(x), Y, u) \end{cases}.$$

Then

1. *the $\mathcal{A}_e$-codimension of the germs g_0 and f_0 is equal to 1, and the germs $\overline{g} : (x,v) \mapsto (g_{v,0}(x), v)$ and $\overline{f} : (y,u) \mapsto (f_u(y), u)$ are $\mathcal{A}_e$- versal unfoldings of g_0 and f_0.*

2. *If also h is primitive, then so are g_0 and f_0.*

Proof We give the proof for g_0 and $\overline{g}$; the proof of f_0 and $\overline{f}$ is identical.
Step 1: The unfolding H of h given by

$$\begin{cases} (X,y,u,v) & \overset{F}{\mapsto} & (X, f_u(y), u+v, v) \\ (x,Y,s,v) & \overset{g\times \mathrm{id}_{\mathbb{C}}}{\mapsto} & (g(x,Y,s), v) \end{cases}$$

is $\mathcal{A}_e$-versal. For it is not infinitesimally trivial, and h has $\mathcal{A}_e$-codimension 1.
Step 2: Let G be an unfolding of g_0, and let $\tilde{G}$ be the direct sum unfolding of G and g. Clearly G can be induced from $\tilde{G}$. Consider the unfolding $\tilde{H}$ of h, given by $\tilde{H} = \{F \times \mathrm{id}_{\mathbb{C}^d}, \tilde{G}\}$. As the 1-parameter unfolding H of h is versal, $\tilde{H}$ must be isomorphic to an unfolding induced from H. This means $\tilde{G}$ is isomorphic to an unfolding induced from $g \times \mathrm{id}_{\mathbb{C}}$. Any such unfolding is isomorphic to an unfolding induced from g. Hence g is a versal unfolding of g_0. The Kodaira-Spencer map of g, from $T_0\mathbb{C}^b \times \mathbb{C}$ to the $\mathcal{A}_e$-normal space of g_0, is therefore surjective. But as g is trivial along $\{0\} \times \mathbb{C}^b \times \{0\}$, the Kodaira- Spencer map is identically zero along $\mathbb{C}^b \times \{0\}$. Hence the restriction of the Kodaira-Spencer map to $\{0\} \times \mathbb{C}$ is surjective, and $\overline{g}$ is $\mathcal{A}_e$-versal.

If also h is primitive, then $\tau(f) \cap \tau(g) = \{0\}$, by 5.12, and so the analytic stratum of the versal unfolding $(x,v) \mapsto (x, g_v(x))$ must be reduced to $\{0\}$ also. It follows from 2.7 that g_0 must be primitive. □

Corollary 5.19 *Suppose that $\{f,g\}$ is a multi-germ of $\mathcal{A}_e$- codimension 1, with f transverse to $\tau(g)$ and g transverse to $\tau(f)$. Then the pull-back of f by $\tau(g)$, and the pullback of g by $\tau(f)$, are both germs of $\mathcal{A}_e$-codimension 1.*

Proof When $\{f,g\}$ is put in the semi-normal form of the Proposition, these pull-backs are just f_0 and g_0, and the proposition establishes that they have $\mathcal{A}_e$-codimension 1. However, the statement is evidently independent of choice of coordinates. □

We would like to be able to put the germ $h = \{f,g\}$ of 5.18 into a normal form,

$$\begin{cases} (f : (X,y,u) \mapsto (X, f_u(y), u) \\ G : (x,Y,v) \mapsto ((g_v(x), Y, v) \end{cases};$$

but it is not clear that this is always possible. The problem is as follows: now that we have established that $(x,u) \mapsto (g_{0,u}(x),u)$ is a versal unfoding of g_0, it follows that there exists a submersion $\gamma : \mathbb{C}^b \times \mathbb{C} \to \mathbb{C}$, and germs of families of diffeomorphisms $\phi_{Y,u}, \psi_{Y,u}$ such that

$$g_{Y,u} = \psi_{Y,u} \circ \gamma_{0,\gamma(Y,u)} \circ \phi_{Y,u};$$

nevertheless, in order to transform h from its semi-normal form to the desired normal form, the $\psi_{Y,u}$ and $\phi_{Y,u}$ would have to satisfy the stronger requirement that

$$g_{Y,u} = \psi_{Y,u} \circ \gamma_{0,u} \circ \phi_{Y,u}.$$

This can be done under certain assumptions of quasihomogeneity, which we now explain.

A map $f : \mathbb{C}^n \to \mathbb{C}^p$ is **weighted homogeneous** if there are positive integers $\omega_1, \ldots, \omega_n$ (the weights) and $d_1, \ldots, d_p$ (the degrees) such that for $\mu \in \mathbb{C}$, $f(\mu^{\omega_1}x_1, \ldots, \mu^{\omega_n}x_n) = (\mu^{d_1}f_1(x), \ldots, \mu^{d_p}f_p(x))$. A germ f is **quasihomogeneous** if it is $\mathcal{A}$-equivalent to a weighted homogeneous map-germ. A multi-germ is quasihomogeneous if its branches are quasihomogeneous with the same degrees.

Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be a quasihomogeneous multi-germ of $\mathcal{A}_e$-codimension 1. When (n,p) are in the range of nice dimensions, we can find a quasihomogeneous versal unfolding $F(\lambda, x) = (\lambda, f_\lambda(x))$ of f such that the degree r of the unfolding parameter is positive. In fact, if the degree is non-positive, then F is topologically trivial and therefore f is topologically stable. But this is a contradition since in the nice dimensions topological stability is equivalent to stability. Let $r, d_1, \ldots, d_p$ be the degrees of the components of F and let $r, w_1^{(i)}, \ldots, w_{n(i)}^{(i)}$ be the weights in the source of the i^{th} branch of F. For $\mu \in \mathbb{C}$ define $\psi_\mu : \mathbb{C}^p \to \mathbb{C}^p$ by $\psi_\mu(y_1, \ldots, y_p) = (\mu^{d_1}y_1, \ldots, \mu^{d_p}y_p)$ and define $\Psi_\mu : \mathbb{C}^{p+1} \to \mathbb{C}^{p+1}$ by $\Psi_\mu(\lambda, y) = (\mu^r\lambda, \psi_\mu(y))$. Let $\phi_\mu^{(i)}$ and $\Phi_\mu^{(i)}$ be the analogues of these maps in the source of the i^{th} branch of f and F respectively. If ϕ_μ has branches $\phi_\mu^{(i)}$ then

$$f_{\mu^r\lambda} \circ \phi_\mu = \psi_\mu \circ f_\lambda$$

Lemma 5.20 *Let $\bar{f}$ and $\tilde{f}$ be quasihomogeneous $\mathcal{A}$-equivalent multi-germs from $\mathbb{C}^n$ to $\mathbb{C}^p$ ((n,p) nice dimensions) of $\mathcal{A}_e$-codimension 1. Let $\bar{F}(\lambda_1, \ldots, \lambda_d, x) = (\lambda_1, \ldots, \lambda_d, \bar{f}_{\lambda_1,\ldots,\lambda_d}(x))$ be a versal unfolding of $\bar{f}$ with analytic stratum $\{0\} \times \mathbb{C}^{d-1} \times \{0\}$ and let $\tilde{F}(\mu, x) = (\mu, \tilde{f}_\mu(x))$ be a versal unfolding of $\tilde{f}$. Then there are families of diffeomorphisms α_λ of $\mathbb{C}^n$ and β_λ of $\mathbb{C}^p$, $\lambda \in \mathbb{C}^d$, such that the following diagram commutes*

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n & \xrightarrow{\bar{F}} & \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p \\ \downarrow \alpha & & \downarrow \beta \\ \mathbb{C} \times \mathbb{C}^n & \xrightarrow{\tilde{F}} & \mathbb{C} \times \mathbb{C}^p \end{array}$$

where $\alpha(\mu, \nu, x) = (\mu, \alpha_{(\mu,\nu)}(x))$ and $\beta(\mu, \nu, y) = (\mu, \beta_{(\mu,\nu)}(y))$.

Proof We may suppose that $\tilde{f}$ and $\tilde{F}$ are quasihomogeneous as maps.

Let ϕ and ψ be diffeomorphisms such that $\psi \circ \bar{f} = \tilde{f} \circ \phi$. Then $F' = (id_{\mathbb{C}^d} \times \psi) \circ \bar{F} \circ (id_{\mathbb{C}^d} \times \phi)^{-1} : \mathbb{C}^d \times \mathbb{C}^n \to \mathbb{C}^d \times \mathbb{C}^p$ is a versal unfolding of $\tilde{f}$ with analytic stratum $\{0\} \times \mathbb{C}^d \times \{0\}$.

Since $\tilde{F}$ is a miniversal unfolding, there is a submersion $\gamma : \mathbb{C}^d \to \mathbb{C}$ and there are families of diffeomorphisms $\bar{\phi}_\lambda$ of $\mathbb{C}^n$ and $\bar{\psi}_\lambda$ of $\mathbb{C}^p$, $\lambda \in \mathbb{C}^d$, such that the following diagram commutes.

$$\begin{array}{ccc}
\mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n & \stackrel{F'}{\to} & \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p \\
\downarrow \Gamma \times \bar{\phi}_\lambda & & \downarrow \Gamma \times \bar{\psi}_\lambda \\
\mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n & \stackrel{\tilde{F} \times id_{\mathbb{C}^{d-1}}}{\to} & \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p
\end{array}$$

where $\Gamma(\mu, \nu) = (\gamma(\mu, \nu), \nu)$.

We have $\gamma^{-1}(0) = \{0\} \times \mathbb{C}^{d-1}$, so Γ is a diffeomorphism by the inverse function theorem. Since Γ commutes with projection onto $\mathbb{C}^{d-1}$, Γ^{-1} does also, so there exists $\gamma', \gamma'' : \mathbb{C}^d \to \mathbb{C}$ such that $\Gamma^{-1}(\mu, \nu) = (\gamma'(\mu, \nu), \nu)$ and $\gamma' = \mu\gamma''$ where $\mu : \mathbb{C} \times \mathbb{C}^{d-1} \to \mathbb{C}$ is the projection onto the first coordinate. Also γ' is a submersion and so γ'' is non-zero in a neighbourhood of the origin. We have

$$(\gamma' \times \psi_{\sqrt[r]{\gamma''}}) \circ (\tilde{F} \times id_{\mathbb{C}^{d-1}}) = \tilde{F} \circ (\gamma' \times \phi_{\sqrt[r]{\gamma''}})$$

where $\phi_{\sqrt[r]{\gamma''}}$ and $\psi_{\sqrt[r]{\gamma''}}$ are as stated just before this proposition. Thus the following diagram commutes

$$\begin{array}{ccc}
\mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^n & \stackrel{F'}{\to} & \mathbb{C} \times \mathbb{C}^{d-1} \times \mathbb{C}^p \\
\downarrow \bar{\alpha} & & \downarrow \bar{\beta} \\
\mathbb{C} \times \mathbb{C}^n & \stackrel{\tilde{F}}{\to} & \mathbb{C} \times \mathbb{C}^p
\end{array}$$

where $\bar{\alpha} = (\gamma' \times \phi_{\sqrt[r]{\gamma''}}) \circ (\Gamma \times \bar{\phi}_\lambda)$ and $\bar{\beta} = ((\gamma' \times \psi_{\sqrt[r]{\gamma''}}) \circ (\Gamma \times \bar{\psi}_\lambda)$.

Now the proposition follows by choosing $\alpha = \bar{\alpha} \circ (id \times \phi)$ and $\beta = \bar{\beta} \circ (id \times \psi)$. □

Now we can continue with the task of reducing a primitive $\mathcal{A}_e$-codimension 1 germ in the semi-normal form

$$\begin{cases} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, v) \mapsto (g_{Y,v}(x), Y, v) \end{cases}$$

to the normal form

$$\begin{cases} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, v) \mapsto (g_v(x), Y, v) \end{cases}.$$

We make the additional hypothesis that g_0 is quasihomogeneous, and is not topologically stable. Then in appropriate coordinates it has an $\mathcal{A}_e$-versal unfolding whose unfolding parameter has positive weight. Thus we can apply 5.20, to deduce that the unfolding g of g_0 is isomorphic to a prism on the unfolding $\overline{g} :\ (x, u) \mapsto (g_{0,u}(x), u)$. That is, there are diffeomorphisms $\Phi : (\mathbb{C}^\alpha \times \mathbb{C}^b \times \mathbb{C}, T) \to (\mathbb{C}^\alpha \times \mathbb{C}^b \times \mathbb{C}, T)$, of the form $\Phi(x, Y, u) = (\phi_{Y,u}(x), Y, u)$, and $\Psi : (\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}, 0) \to (\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}, 0)$ of the form $\Psi(X, Y, u) =$

$(\psi_{Y,u}(X), Y, u)$, such that $g_{Y,u}(x) = \psi_{Y,u} \circ g_{0,u} \circ \phi_{Y,u}$. Composing with Φ in the source of g, and Ψ on the target of $\{f, g\}$, we bring $\{f, g\}$ to the form

$$\begin{cases} (X, y, u) \mapsto (\psi_{f_u(y),u}(X), f_u(y), u) \\ (x, Y, v) \mapsto (g_{0,v}(x), Y, v) \end{cases}$$

and now if we take the first a coordinates of $\Psi \circ f$ as new coordinates on the domain of f, we bring $\{f, g\}$ to the desired normal form. We have proved

Theorem 5.21 *If $h = \{f, g\}$ is a multi-germ of $\mathcal{A}_e$ codimension 1, in which f is transverse to $\tau(g)$ and g is transverse to $\tau(f)$, and if either the pullback of f by $\tau(g)$ or the pullback of g by $\tau(f)$ is quasihomogeneous and not topologically stable, then $\{f, g\}$ is equivalent to a binary concatenation $B(f_0, g_0)$; that is, to a germ of the form*

$$\begin{cases} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, v) \mapsto (g_v(x), Y, v) \end{cases}.$$

□

We now summarise the results of this section:

Theorem 5.22 *Let $h = \{f, g\}$ be a primitive $\mathcal{A}_e$-codimension 1 map-germ in the nice dimensions (with no submersive branches). Then f and g are both stable (5.7).*

1. *If f and g are not transverse, then (5.16) h is equivalent to*

$$\begin{cases} (x_1, \ldots, x_n) \mapsto \sum_i x_i^2 \\ (y_1, \ldots, y_m) \mapsto \sum y_j^2. \end{cases}$$

   Now assume $f \pitchfork g$.

2. *If g is not transverse to $\tau(f)$, then (5.16) f is transverse to $\tau(g)$, and h is equivalent to*

$$\begin{cases} (x_1, \ldots, x_n, u) \mapsto (f_u(x), u) \\ (\lambda_1, \ldots, \lambda_{p-1}, v_1, \ldots, v_k) \mapsto (\lambda, \sum_i v_i^2) \end{cases}$$

   (so $\{f, g\}$ is equivalent to $C_k(f_0)$).

3. *If $g \pitchfork \tau(f)$ and $f \pitchfork \tau(g)$, then (5.21) $\{f, g\}$ is equivalent to a germ of the form*

$$\begin{cases} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, v) \mapsto (g_{Y,v}(x), Y, v) \end{cases}$$

   where the target is decomposed as $\mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$, and f_0 and g_0 are primitive. If also the pullback of g by $\tau(f)$ or the pullback of f by $\tau(g)$ is quasihomogeneous then $\{f, g\}$ is equivalent to

$$\begin{cases} (X, y, u) \mapsto (X, f_u(y), u) \\ (x, Y, v) \mapsto (g_v(x), Y, v), \end{cases}$$

   i.e. to $B(f_0, g_0)$.

□

Remark 5.23 If we replace $\mathbb{C}$ by $\mathbb{R}$ and analytic maps by smooth ones, then the results obtained so far still hold modulo the following alterations: in the real case we define two augmentations: $A_F^+(\lambda, x) = (\lambda, f_{\lambda^2}(x))$ and $A_F^-(\lambda, x) = (\lambda, f_{-\lambda^2}(x))$. In the proof of Proposition 5.20, if ω is even then we cannot necessarily define $\sqrt[\omega]{\ }$ properly. Consequently we may have to define $\alpha(\mu, \nu, x) = (-\mu, \alpha_{(\mu,\nu)}(x))$ and $\beta(\mu, \nu, y) = (-\mu, \beta_{(\mu,\nu)}(y))$ in order to get the diagram to commute.

6 Topology

Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ $(n \geq p-1,\ (n,p)$ nice dimensions and S a finite set) be a finitely $\mathcal{A}$-determined multi-germ. A **stabilisation** of f is a 1-parameter unfolding $F : (\mathbb{C} \times \mathbb{C}^n, \{0\} \times S) \to (\mathbb{C} \times \mathbb{C}^p, (0,0))$ with the property that there is a representative $F : U \to V$ (we shall use the same letter) and a positive real number δ such that for $\lambda \in B_\delta(0) \setminus \{0\}$, the map $f_\lambda : U_\lambda \to V_\lambda$ is infinitesimally stable (here $U_\lambda = U \cap (\{\lambda\} \times \mathbb{C}^n)$ and $V_\lambda = V \cap (\{\lambda\} \times \mathbb{C}^p)$), $F|_{\sum(F)}$ is proper, finite to one and generically one to one, and that $F^{-1}(0,0) \cap \sum(F) = \{0\} \times S$. It follows that the discriminant $D(F)$ of F is a closed analytic subset of V. The mapping f_λ is a **stable perturbation** of f.

Consider the canonical stratification of $D(F)$ and choose $\epsilon > 0$ such that for all ϵ' with $0 < \epsilon' \leq \epsilon$, $D(f) \cong D(F) \cap (\{0\} \times \mathbb{C}^p)$ is stratified transverse to the sphere $S_{\epsilon'} \subset \mathbb{C}^p$ of centre 0 and radius ϵ'. Such ϵ is called a **Milnor radius** for $D(f)$. By Thom's First Isotopy Lemma, $D(f) \cap B_\epsilon$ is a cone on its boundary $D(f) \cap S_\epsilon$. It follows that there is a $\delta > 0$ such that for $\lambda \in B_\delta \subseteq \mathbb{C}$, $D(F)$ is stratified transverse to $\{\lambda\} \times S_\epsilon$ (we call such a δ a **perturbation limit** for F with respect to B_ϵ). For $\lambda \in B_\delta$, the **discriminant** of f_λ is defined to be $D(f_\lambda) \cap B_\epsilon$, or, in other words, $D(F) \cap (\{\lambda\} \times B_\epsilon)$.

For $\epsilon_1, \ldots, \epsilon_p > 0$ define the set $P_{\epsilon_1,\ldots,\epsilon_p}(0)$ to be the polycylinder $\{(y_1, \ldots, y_p) \in \mathbb{C}^p \,/\, |y_i| < \epsilon_i \ \forall i\}$. We shall also use the term "Milnor radius for $D(f)$" for an $\epsilon > 0$ such that for all $\epsilon_1, \ldots, \epsilon_p$ with $0 < \epsilon_i < \epsilon$ $(\forall i)$, $D(f)$ is stratified transverse to the boundary of the polycylinder $P_{\epsilon_1,\ldots,\epsilon_p}(0)$. The results described above apply with such a polycylinder replacing B_ϵ and the discriminant defined this way is the same.

Let $\pi : D(F) \to \mathbb{C}$ be the projection to the parameter space $\mathbb{C}$. It follows by [4] that π induces a locally trivial fibration

$$((B_\delta \backslash \{0\}) \times B_\epsilon) \cap D(F) \to B_\delta \backslash \{0\}$$

Lemma 6.1 *Let A, B be contractible open subsets of a topological space X, and A', B' be contractible open subsets of X'. Suppose that $A \cap B$ and $A' \cap B'$ are homotopy equivalent, and moreover that $A \cap B$ has collared neighbourhoods in both A and B, and $A' \cap B'$ has collared neighbourhoods in both A' and B'. Then $A \cup B$ and $A' \cup B'$ are homotopy equivalent.*

□

Suppose f has $\mathcal{A}_e$-codimension 1. Let $F(\lambda, x) = (\lambda, f_\lambda(x))$ be a proper representative of a miniversal unfolding of f. For $\mu \in \mathbb{C}$ define $g_\mu(\lambda, x) = (\lambda, f_{\lambda^2+\mu}(x))$. Then $G(\mu, \lambda, x) = (\mu, \lambda, f_{\lambda^2+\mu}(x))$ is a proper representative of a miniversal unfolding of $g = A_F f$.

Theorem 6.2 *With the above notation, for $\mu \neq 0 \neq \lambda$ the discriminant of g_μ is homotopy equivalent to the suspension of the discriminant of f_λ.*

Proof Let $\epsilon > 0$ be a Milnor radius for both f and F, also let $\delta > 0$ be a perturbation limit for F with respect to $P_{\epsilon,\ldots,\epsilon}(0) \subseteq \mathbb{C}^p$.

Let $\epsilon' > 0$ be a Milnor radius for g and let $\delta' > 0$ be a perturbation limit for G with respect to $P_{\epsilon'',\epsilon,\ldots,\epsilon}(0) \subseteq \mathbb{C}^{p+1}$, where $\epsilon'' = min\{\epsilon, \sqrt{\delta/2}\}$.

Fix $\mu_0 \in \mathbb{C}$ and consider

$$\pi : D(g_{\mu_0}) \cap P_{\epsilon'',\epsilon,\ldots,\epsilon}(0) \to \mathbb{C}$$

be the projection onto the first coordinate. For a convenient choice of μ_0 we have

(i) the fibre of π over $\lambda \in B_{\epsilon''}(0)$ is naturally homeomorphic to $D(f_{\lambda^2+\mu_0}) \cap P_{\epsilon,\ldots,\epsilon}(0)$ which is the discriminant of $f_{\lambda^2+\mu_0}$.

(ii) Suppose that the square roots of $-\mu_0$ are in $B_{\epsilon''}(0)$, say a and b. Then the restriction of π to $\pi^{-1}(B_{\epsilon''}(0) \backslash \{a, b\})$ is a locally trivial fibration.

Let A and B be contractible open subsets of $B_{\epsilon''}(0)$ with contractible (non-empty) intersection such that $a \in A \backslash B$ and $b \in B \backslash A$.

By standard arguments we can conclude that $\pi^{-1}(A \cup B)$ is homotopy equivalent to the discriminant of g_{μ_0} and $\pi^{-1}(A \cap B)$ is homotopy equivalent to the discriminant of f_λ; we can also assume that $\pi^{-1}(A \cap B)$ is collared in both $\pi^{-1}(A)$ and $\pi^{-1}(B)$. Since the suspension of any space D can be divided into two contractible subspaces whose intersection has collared neighbourhoods and is homotopy equivalent to D, by Lemma 6.1 we have only to prove that $\pi^{-1}(A)$ and $\pi^{-1}(B)$ are contractible. At a, $\gamma(\lambda) = \lambda^2 + \mu_0$ is a diffeomorphism and induces a homeomorphism between $\pi^{-1}(\gamma^{-1}(B_{\delta''}(0)))$ and $D(F) \cap P_{\delta'',\epsilon,\ldots,\epsilon}(0)$ for some $\delta'' > 0$. Therefore $\pi^{-1}(A)$ is contractible since it is homeomorphic to $\pi^{-1}(\gamma^{-1}(B_{\delta''}(0)))$ and $D(F) \cap P_{\delta'',\epsilon,\ldots,\epsilon}(0)$ is a cone. Similarly, $\pi^{-1}(B)$ is contractible. $\square$

We now determine the homotopy-type of the discriminant of a stable perturbation of a concatenation.

Proposition 6.3 *Let f_0 be a multi-germ of finite $\mathcal{A}_e$-codimension, which has a 1-parameter stable unfolding F. The discriminant of a stable perturbation of the multi-germ $C_k(f_0)$ (i.e. $\{F, g\}$, where $g(y, v) = (y, \sum v_i^2)$) is homotopy-equivalent to the suspension of the discriminant of a stable perturbation of f_0.*

Proof A stable perturbation h_λ of h has branches F and $g_\lambda(y,v) = (y, \sum v_i^2 + \lambda)$. The discriminant of h_λ is the union of two contractible spaces: the discriminant of F and the discriminant of g_λ. The intersection of these sets is the discriminant of $\tilde{f}_\mu$, which is a stable perturbation of $\tilde{f}_0$. The proposition now follows from 6.1 in the same way as Theorem 6.2. □

In order to deal with the discriminant of a binary concatenation $B(f_0, g_0)$, we need some topological results.

Let X and Y be topological spaces. The **join** of X and Y, $X * Y$, is the space $(X \times Y \times I)/\sim$ where $(x, y, \lambda) \sim (x', y', \lambda')$ if and only if either $\lambda = \lambda' = 0$ and $y = y'$ or $\lambda = \lambda' = 1$ and $x = x'$.

Lemma 6.4 *If X_1 is homotopy equivalent to X_2 and Y_1 is homotopy equivalent to Y_2 then $X_1 * Y_1$ is homotopy equivalent to $X_2 * Y_2$.* □

Corollary 6.5 *If X_1 is homotopy equivalent to X_2 then $S(X_1)$ is homotopy equivalent to $S(X_2)$.* □

Proposition 6.6 *Suppose that $h = B(f_0, g_0)$ is a binary concatenation,*

$$\begin{cases} (X, y, u) & \overset{f}{\mapsto} & (X, f_u(y), u) \\ (x, Y, v) & \overset{g}{\mapsto} & (g_v(x), Y, v) \end{cases}$$

of germs f_0 and g_0 of finite codimension, as described in Theorem 3.8. Let H be the stable unfolding of h given by

$$\begin{cases} (X, y, u, t) & \overset{F}{\mapsto} & (X, f_u(y), u + t, t) \\ (x, Y, v, t) & \overset{g \times id_{\mathbb{C}}}{\mapsto} & (g_v(x), Y, v, t) \end{cases}.$$

Then for $t \neq 0$ the discriminant of the stable perturbation h_t of h is homotopy equivalent to the suspension of $D(\tilde{f}_{-t}) * D(\tilde{g}_t)$, and thus $\mu_\Delta(h) = \mu_\Delta(\tilde{f}_0) \times \mu_\Delta(\tilde{g}_0)$.

Proof The discriminant of h_t is the union of the (contractible) discriminants of $(X, y, u) \mapsto (X, f_u(y), u + t)$ and $(X, y, v) \mapsto (g_v(x), Y, v)$. It is prefereble to re-parametrise the first as the image of $(X, y, u) \mapsto (X, f_{u-t}(y), u)$. Call these two spaces D_1 and D_2. By 6.1, $D_1 \cup D_2$ is homotopy-equivalent to the suspension of $D_1 \cap D_2$. Let $\epsilon > 0$ be a Milnor radius for f_0 and g_0, and let $P_f = P_{\epsilon,\ldots,\epsilon}(0) \subset \mathbb{C}^b$ and $P_g = P_{\epsilon,\ldots,\epsilon}(0) \subseteq \mathbb{C}^a$. Thus, we have to show that inside a suitable Milnor polycyclinder $P_f \times P_g \times B(0, \epsilon') \subset \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C}$, and for $0 < |t| < \delta << \epsilon$, $D_1 \cap D_2$ is homotopy-equivalent to the join of $D(f_{-t}) \cap P_f$ and $D(g_t) \cap P_g$. This follows by a standard argument from the following three facts:

1. The projection $\pi_h : \mathbb{C}^a \times \mathbb{C}^b \times \mathbb{C} \to \mathbb{C}$ induces a locally trivial fibration
$$D_1 \cap D_2 \cap \pi_h^{-1}(B_\delta \backslash \{0,t\}) \to B_\delta \backslash \{0,t\}$$
whose fibre is homotopy equivalent to $D(g_t) \times D(f_{-t})$.

2. The fibre of π_h over t is $D(g_t) \times D(f_0)$; because $D(f_0)$ is contractible, this is homotopy-equivalent to $D(g_t)$.

3. The fibre of π_h over 0 is $D(g_0) \times D(f_{-t})$; because $D(g_0)$ is contractible, this is homotopy-equivalent to $D(f_{-t})$.

Let $[0,t]$ denote the line-segment joining 0 and t in $\mathbb{C}$. Clearly there is a deformation-retraction $B_\delta \to [0,t]$; since π_h is locally trivial on the complement of $[0,t]$, this lifts to a deformation-retraction $D = \pi_h^{-1}(B_\delta) \to \pi^{-1}([0,t]$. By (1),(2),(3) above, $\pi^{-1}([0,t]$ is homotopy-equivalent to $D(g_t) * D(f_{-t})$. □

Now we consider the real case. Let $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ ($n \geq p-1$, (n,p) nice dimensions) be a multi-germ of $\mathcal{A}_e$-codimension 1 and let $F(\lambda, x) = (\lambda, f_\lambda(x))$ be a miniversal unfolding. Up to homeomorphism, there are two (possibly equivalent) choices for the discriminant of f_λ: one with positive λ and one with negative λ. We shall call these $D^+(f)$ and $D^-(f)$ respectively. Recall from 5.23 that in the real case, f has two augmentations $g = A_F^+ f$ and $\tilde{g} = A_F^- f$ with stable perturbations $g_\mu(\lambda, x) = (\lambda, f_{\lambda^2+\mu}(x))$ and $\tilde{g}_\mu(\lambda, x) = (\lambda, f_{-\lambda^2+\mu}(x))$ respectively.

Proposition 6.7 *With the above notation*

$$(i) D^+(g) \cong D^+(f) \qquad (ii) D^-(g) \cong S(D^-(f))$$

$$(iii) D^+(\tilde{g}) \cong S(D^+(f)) \qquad (iv) D^-(\tilde{g}) \cong D^-(f)$$

In particular, if f has a good real perturbation then so does one of its two augmentations.

Proof By symmetry it is sufficient to show just the first two homotopy equivalences. Case (ii) is analogous to Theorem 6.2 but if we follow the same proof in (i), then since $-\mu_0$ has no real square roots, $D^+(g_\mu)$ is a fibre bundle over $B_{\epsilon''}(0)$ with fibre $D^+(f_\lambda)$. But the total space of a bundle over a contractible space is homotopy equivalent to the fibre. □

We now describe the topology of a discriminant of a stable perturbation over $\mathbb{R}$ of a real germ in the normal forms of Theorem 5.22 (see Remark 5.23).

Proposition 6.3 holds in a slightly different version. Here we have to consider the two discriminants of a stable perturbation of h as well as the two discriminants of a stable perturbation of f_0. We leave the straightforward details to the reader, although we recall that in 3.2 we have already shown that if f_0 has a good real perturbation then so does $C_k(f_0)$. Finally, although we have made no attempt to determine the number of inequivalent real forms of a binary concatenations of two real $\mathcal{A}_e$-codimension 1 multi-germs, the proof of Proposition 6.6 shows

Proposition 6.8 *Suppose $h = B(f_0, g_0)$ is a binary concatenation of two real multi-germs. Then the discriminant of a stable perturbation of h (over $\mathbb{R}$) is homotopy-equivalent to one of the following four spaces:*

$$S(D^+(f_0) * D^+(g_0)) \qquad S(D^-(f_0) * D^+(g_0))$$

$$S(D^+(f_0) * D^-(g_0)) \qquad S(D^-(f_0) * D^-(g_0)).$$

In particular, if f_0 and g_0 have good real perturbations, then so does at least one real form of $B(f_0, g_0)$. □

Example 6.9 Consider the bi-germ consisting of two prisms on Whitney cusps, each transverse to the analytic stratum of the other:

$$h : \begin{cases} f(\lambda, x, \mu) = (\lambda, x^3 + \mu x, \mu) \\ g(z, \delta, \mu) = (z^3 - \mu z, \delta, \mu) \end{cases}$$

The discriminant of each is the product with a line of a plane first-order cusp. The real discriminant of a stable perturbation h_t of h (in which t is added to the third component of g) is thus the union of two prisms, drawn with dotted lines in Figure 5 on page; its homology is carried by the curvilinear tetrahedron drawn with a solid line.

The intersection of $D(h_t)$ with the horizontal plane L_μ, for $0 < \mu < t$, is the union of two pairs of parallel lines, $\mathbb{R} \times D(\tilde{f}_\mu)$ and $D(\tilde{g}_\mu) \times \mathbb{R}$ (since each of $D(\tilde{f}_\mu)$ and $D(\tilde{g}_\mu)$ consists just of a pair of points). $L_\mu \cap D(h_\mu)$ retracts to a rectangle, the intersection of L_μ with the (boundary of the) curvilinear tetrahedron. This rectangle is the join of $D(\tilde{f}_\mu)$ and $D(\tilde{g}_\mu)$.

7 Proofs of the main theorems

Theorem 7.1 *Let $h : (\mathbb{C}^n, T) \to (\mathbb{C}^p, 0)$ ($n \geq p-1$, (n,p) nice dimensions) be a multi-germ of $\mathcal{A}_e$-codimension 1 and corank 1. Then h is quasihomogeneous.*

Proof We may suppose h primitive and ignore any submersive branches. The proof is by induction on the number, $|T|$, of components of h.

If $|T| = 1$, h is quasihomogeneous by results of Victor Goryunov in [5] when $n \geq p$ and by our Proposition 4.3 when $p = n + 1$.

Suppose $h = \{f, g\}$ has more than one branch. If g is not transverse to $\tau(f)$, then by 5.22, either f and g are both prisms on Morse singularities, or h is equivalent to $C_k(f_0)$ for some $\mathcal{A}_e$-codimension 1 germ f_0. In the first case h is plainly quasihomogeneous. In the second, we apply the inductive hypothsis to conclude that f_0 is quasihomogeneous. Since we are in the nice dimensions, f_0 has a quasihomogeneous versal unfolding $\tilde{f}$, and by 5.16, h is equivalent to $C_k(f_0)$. Clearly this is quasihomogeneous.

If f is transverse to $\tau(g)$ and vice versa, then by 5.19 the pullback f_0 of f by $\tau(g)$, and the pull-back g_0 of g by $\tau(f)$, both have codimension 1. By the induction hypothesis, f_0 and g_0 are both quasihomogeneous. By 5.21, $\{f, g\}$ is equivalent to $B(f_0, g_0)$; again, as we are in the nice dimensions, f_0 and g_0 have weighted homogeneous $\mathcal{A}_e$-versal unfoldings with unfolding parameter with positive weight; a representative of $B(f_0, g_0)$ constructed from these ingredients is evidently weighted homogeneous. $\square$

In the next result, we do not distinguish between μ_I and μ_Δ, for the reasons described at the start of Section 3.

Theorem 7.2 *If $h : (\mathbb{C}^n, T) \to (\mathbb{C}^p, 0)$ ($n \geq p - 1$, n, p nice dimensions) has corank 1 and $\mathcal{A}_e$-codimension 1 then $\mu_\Delta(h) = 1$ (and in particular $\mu_I = 1$ for pure-dimensional multi-germs $\mathbb{C}^n \to \mathbb{C}^{n+1}$).*

Proof The proof follows exactly the same scheme as the preceding proof. The starting point for the induction is now the fact that mono-germs of $\mathcal{A}_e$-codimension 1 have μ_I or μ_Δ equal to 1, by our proposition 4.4 for $n = p - 1$, and by the fact (proved in [4]) that $\mu_\Delta = \mathcal{A}_e$-codimension in the nice dimensions, for quasi-homogeneous germs $\mathbb{C}^n \to \mathbb{C}^p$ with $n \geq p$.

We may suppose h primitive; for by 6.2 $D((Ah)_t) \simeq S(D(h_t))$, where the suffix t indicates stable perturbation and S is suspension.

Since the result is already proven in case all branches have $n \geq p$, we assume at least one branch has $n = p-1$. Hence by induction and 5.22 h is equivalent either to $C_k(f_0)$ or to $B(f_0, g_0)$, where f_0 and g_0 are quasihomogeneous $\mathcal{A}_e$-codimension 1 germs. The conclusion now follows by Theorem 3.1(2) for $C_k(f_0)$ and by Theorem 3.8(3) for $B(f_0, g_0)$. $\square$

Theorem 7.3 *Let $h : (\mathbb{C}^n, T) \to (\mathbb{C}^p, 0)$ ($n \geq p - 1$, (n, p) nice dimensions) be a multi-germ of $\mathcal{A}_e$-codimension 1 and corank 1. Then there exists a real form with a good real perturbation.*

Proof Again, the proof is by induction on $|T|$. The result is proven for mono-germs in [19] (for $n \geq p$) and in 4.4 above for the case $p = n + 1$. The inductive steps follow, using the classification theorem 5.22, by 3.2, 3.13 and 6.8. $\square$

References

[1] N.A'Campo, *Le groupe de monodromie du déploiment des singularitiés isolées de courbes planes I*, Math Annalen 213 (1975), 1-31.

[2] T.Cooper, *Mapgerms of $\mathcal{A}_e$-codimension one*, Ph. D. Thesis, University of Warwick, 1994.

[3] J.Damon, *$\mathcal{A}$-equivalence and equivalence of sections of images and discriminants*, Springer Lecture Notes in Mathematics 1462 (1991), 93-121.

[4] J.Damon and D.Mond, *$\mathcal{A}$-codimension and the vanishing topology of discriminants*, Invent. Math. 106 (1991), 217-242.

[5] V.V. Goryunov, *Singularities of projections of full intersections*, Journal of Soviet Mathematics 27, (1984) 2785-2811

[6] V.V. Goryunov, *Monodromy of the image of a mapping*, Functional Analysis and Applications 25 (1991), 174-180

[7] V.V.Goryunov and D.Mond, *Vanishing cohomology of singularities of mappings*, Compositio Mathematica 89 (1993), 45-80.

[8] S.M.Gussein-Zade, *Dynkin diagrams for certain singularities of functions of two real variables*, Functional Analysis and Appl. 8 (1974), 295-300.

[9] K.Houston, *On singularities of folding maps and augmentations*, to appear, Math. Scandinavica.

[10] K.Houston and N.Kirk, *On the classification and geometry of corank 1 map-germs from three-space to four-space*, in J.W.Bruce and D. Mond (eds.) *Singularity Theory*, London Math. Soc. Lecture Notes 263, Cambridge University Press, 1999, 325-351

[11] T. de Jong and D. van Straten, *Disentanglements*, in Singularity Theory and Applications, Warwick 1989, (Lecture Notes in Math. vol 1462), Springer Verlag, 1991, 199-211

[12] E.J.N.Looijenga, *Isolated singular points on complete intersections*, London Math. Soc. LNS 77, Cambridge University Press, 1984

[13] W.L.Marar and D.Mond, *Multiple point schemes for corank 1 maps*, Jour. London Math. Soc. 39 (1989), 553-567.

[14] W.L.Marar and D.Mond, *Real map-germs with good real perturbations*, Topology 35 (1996), 157-165.

[15] J. Martinet, *Singularities of Smooth Functions and Maps*, London Math. Soc. Lecture Notes 58, Cambridge University Press, 1982

[16] J.N.Mather, *Stability of $\mathbb{C}^\infty$ mappings IV: Classification of stable germs by $\mathbb{R}$-algebras*, Pub. Math. IHES 37 (1969), 523-548.

[17] J.N.Mather, *Stability of $\mathbb{C}^\infty$ mappings VI: The nice dimensions*, in C.T.C.Wall (ed.)*Proceedings of the Liverpool Singularities Symposium I*, Lecture Notes in Math. 192, Springer Verlag (1970), pp. 207-253

[18] D.Mond, *On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$*, Proc. London Math. Soc. (3) 50 (1985) pp. 333-369

[19] D.Mond, *How good are real pictures?*, Progress in Maths.134, Birkhauser-Verlag (1996), 259-276.

[20] D.Mond, *Vanishing cycles for analytic maps*, Springer Lecture Notes in Mathematics 1462 (1991), 221-234.

[21] D.Mond, *Looking at bent wires: $\mathcal{A}_e$-codimension and the vanishing topology of parametrized curve singularities*, Math.Proc. Camb.Phil. Soc. 117 (1995), 213-222.

[22] D.Mond, *Differential forms on free and almost free divisors*, preprint, Warwick, 1998

[23] P.Orlik and L.Solomon, *Singularities II: Automorphisms of Forms*, Math. Ann. 231 (1978), 229-240.

[24] A.A. du Plessis, *On the determinacy of smooth map-germs*, Invent. Math. 58 (1980), 107-160

[25] C.T.C.Wall, *A note on symmetry of singularities*, Bull. London Math. Soc. 12 (1980), 169-175.

[26] R.Wik Atique, *On the classification of multi-germs of maps from $\mathbb{C}^2$ to $\mathbb{C}^3$ under $\mathcal{A}$-equivalence*, in J.W.Bruce and F.Tari (eds.) *Real and Complex Singularities*, Research Notes in Maths Series, Chapman & Hall / CRC, 119-133.

Author addresses

Thomas Cooper
Mathematics Institute
University of Warwick
Coventry CV4 7AL
United Kingdom

David Mond
Mathematics Institute
University of Warwick
Coventry CV4 7AL
United Kingdom

mond@maths.warwick.ac.uk

Roberta Wik Atique
Instituto de Ciências Matemâticas
e de Computação
Caixa Postal 668, Sao Carlos, SP
CEP 13560-970, Brazil

rwik@icmc.sc.usp.br